



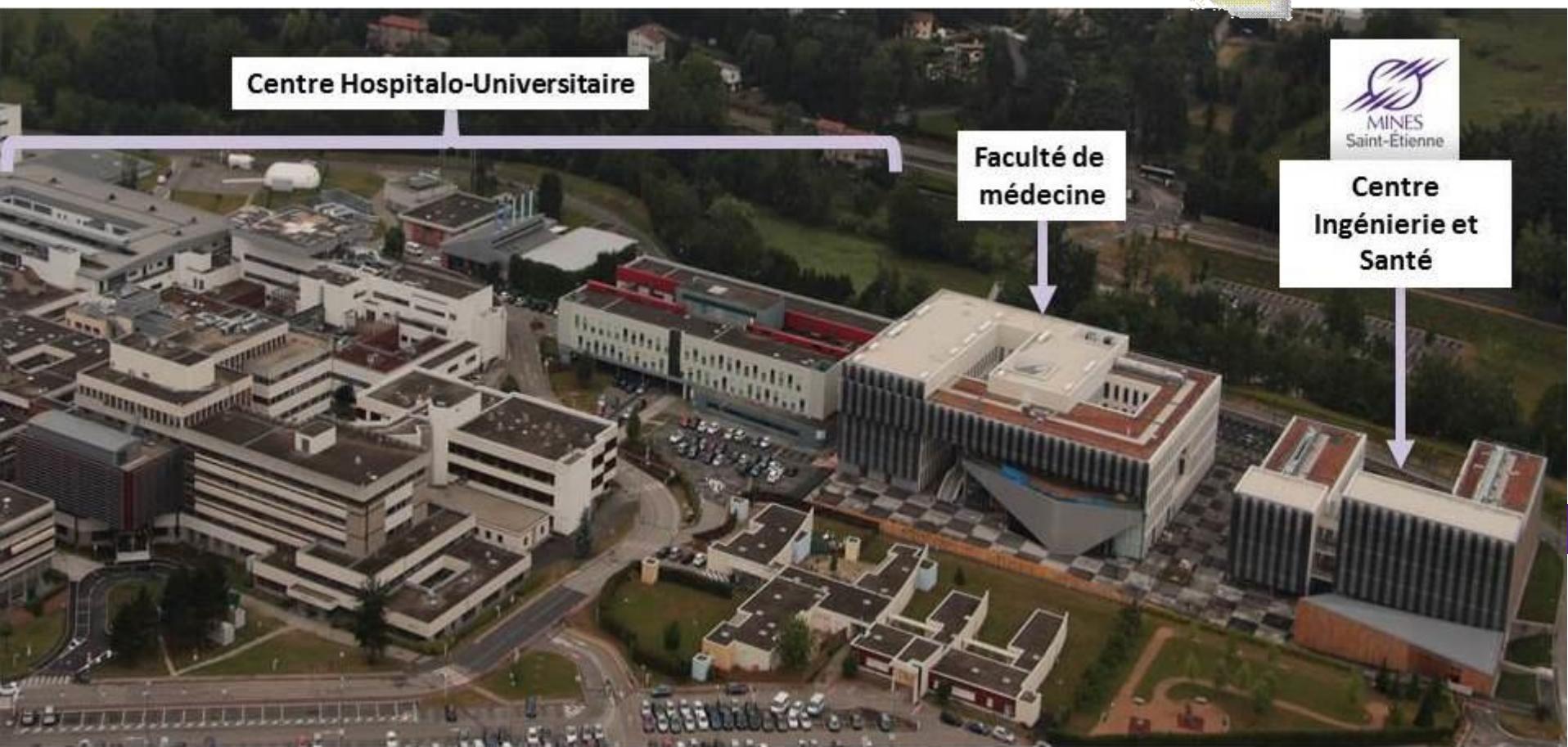
Recent advances in the Virtual Fields Method for Nonlinear Elasticity



Dr. Yue Mei, Prof. Stéphane AVRIL



MINES SAINT-ETIENNE
First Grande Ecole
outside Paris
Founded in 1816

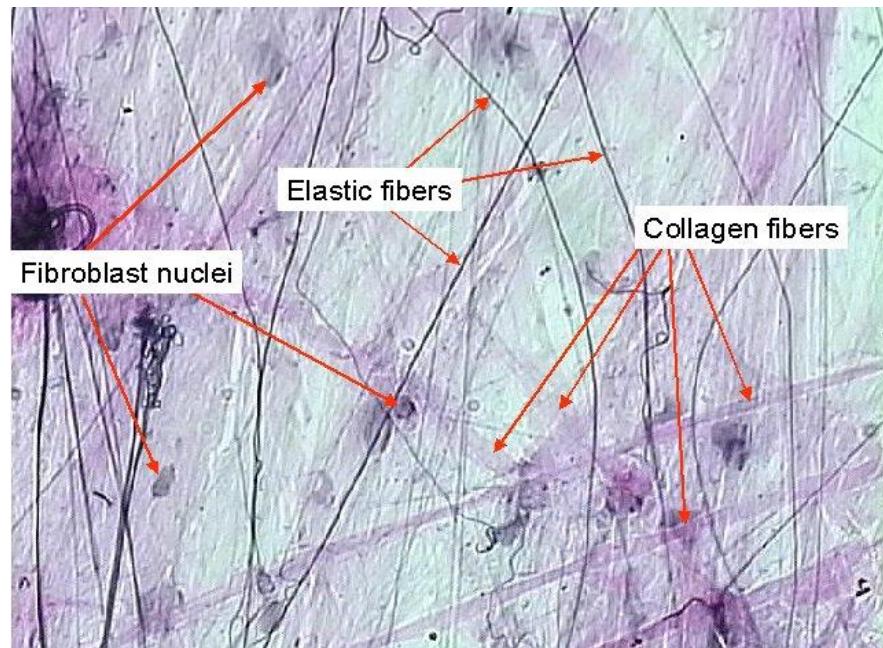


**Centre
Ingénierie et
Santé**



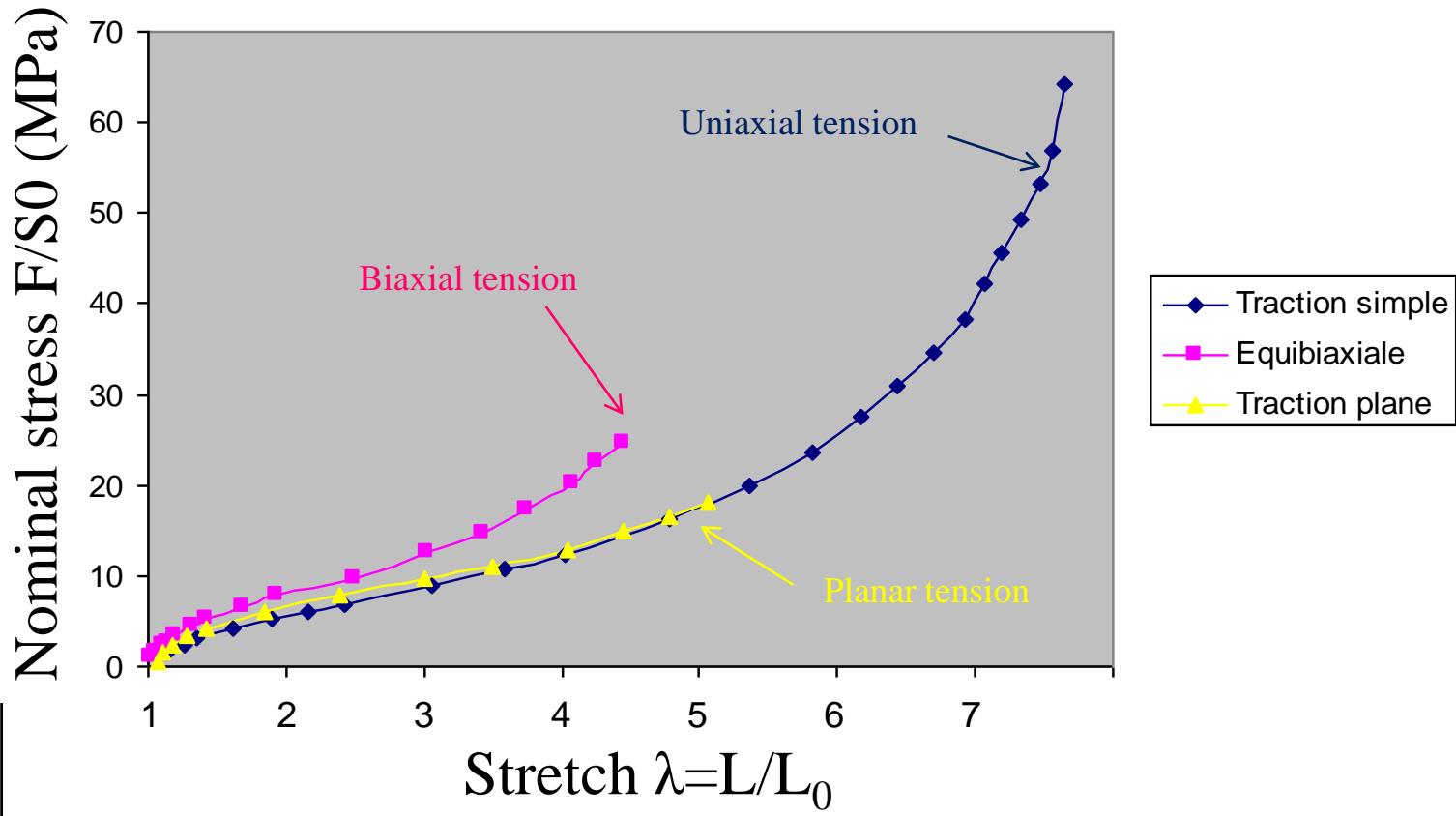
Non linear elasticity

Soft biological tissues: many challenges for continuum mechanics

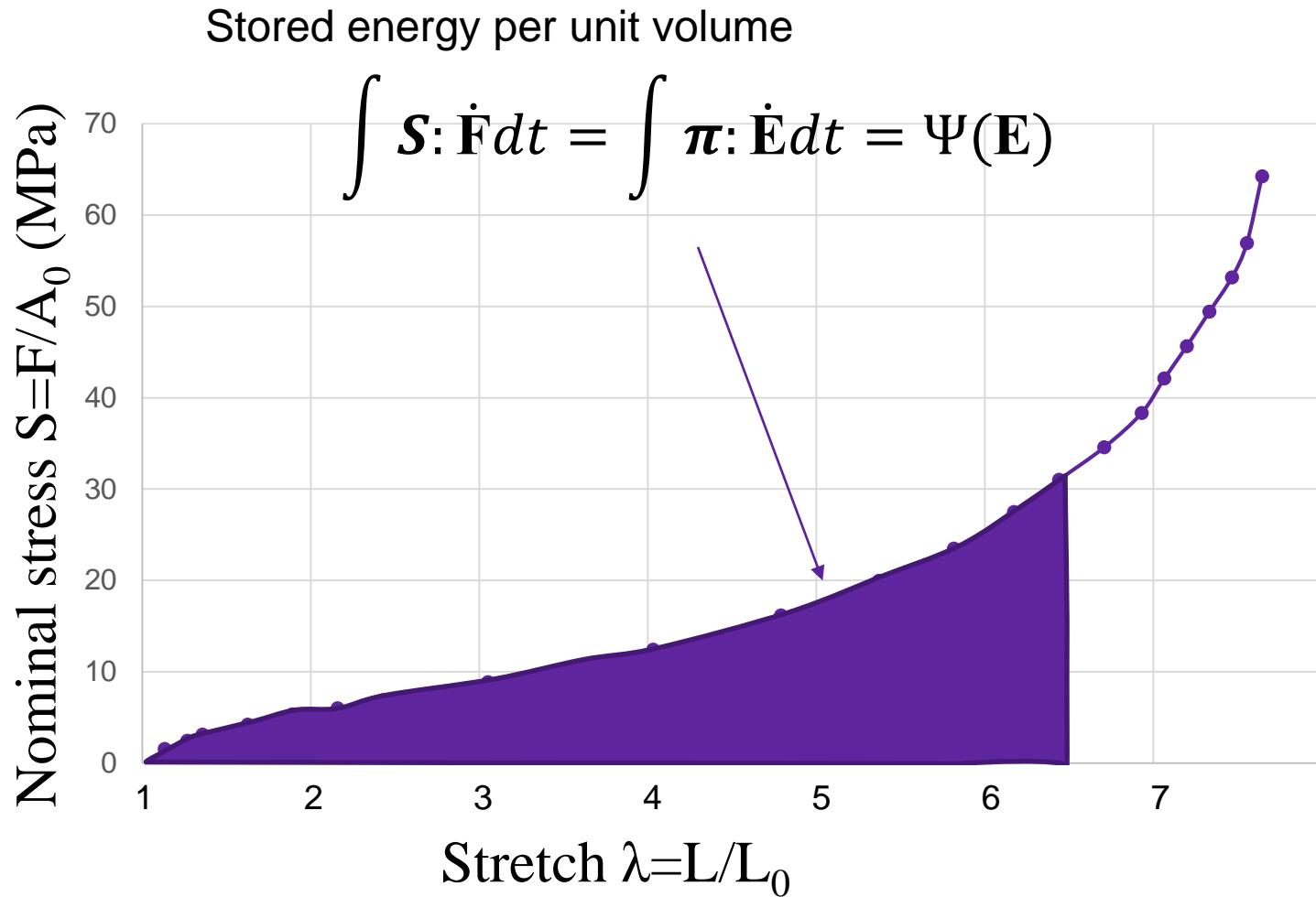


Hyperelasticity

Treolar's tests on rubber (1944)



Strain energy density function





compressible hyperelastic behaviour

$$\boldsymbol{\sigma} = J \mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{E}} \cdot \mathbf{F}^T$$

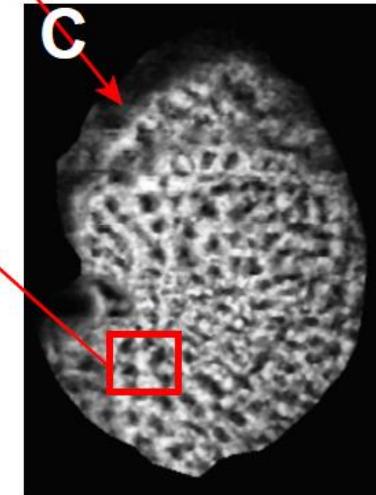
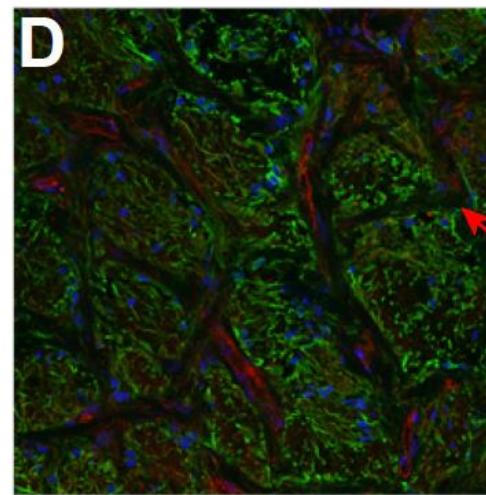
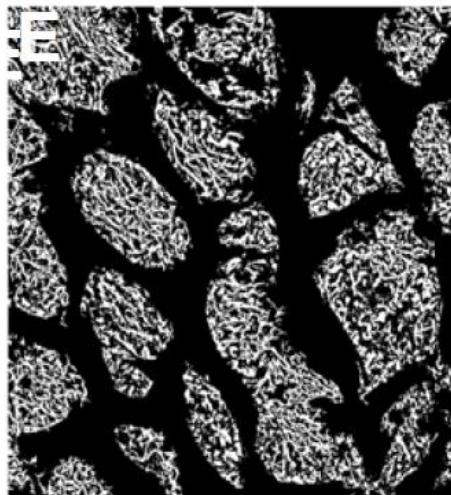
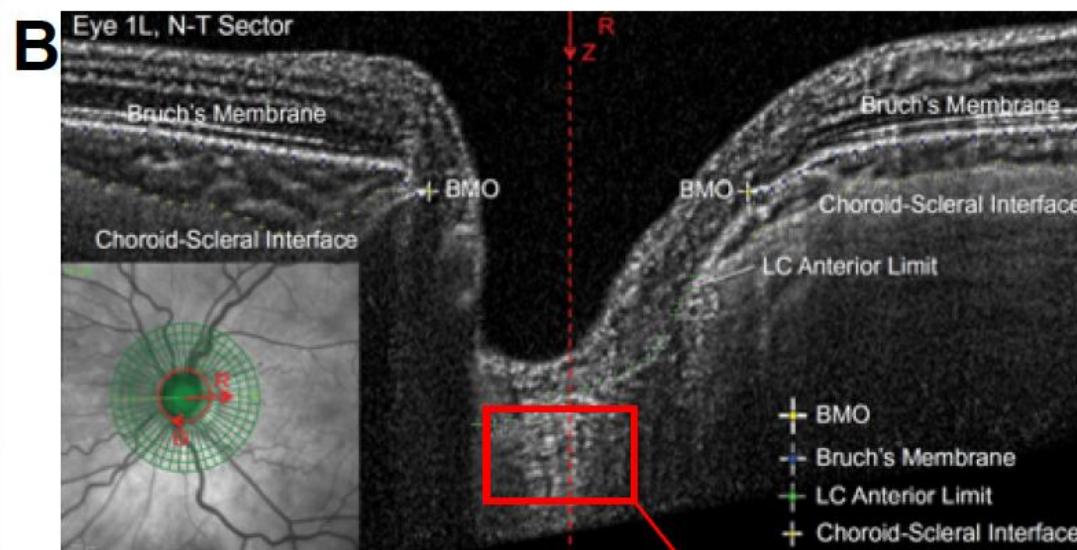
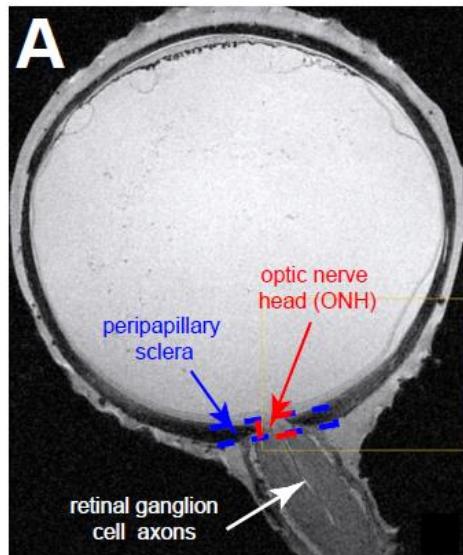
incompressible hyperelastic behaviour ($J=1$)

$$\boldsymbol{\sigma} = \mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{E}} \cdot \mathbf{F}^T + c \mathbf{I}$$

Strain energy density:

$$\Psi = ?$$

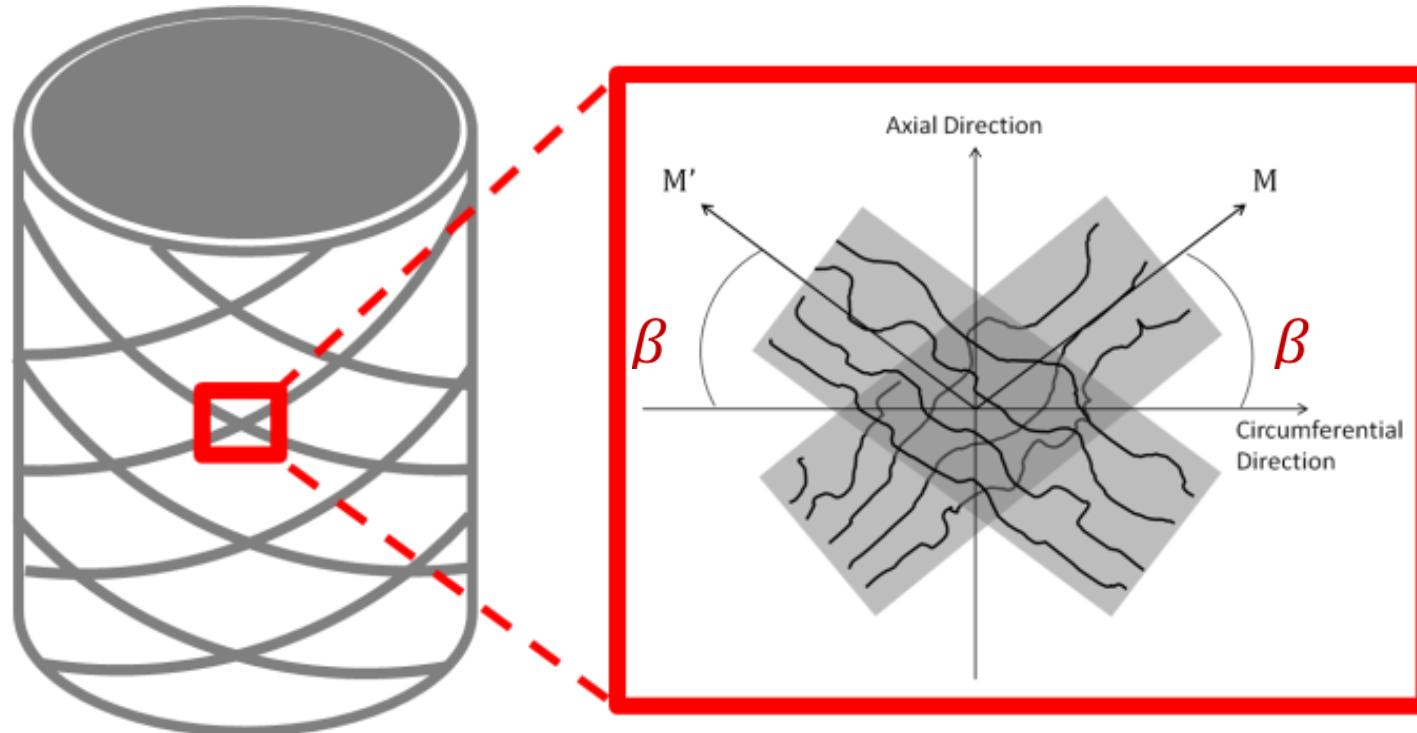
Specific problem of the optical nerve head



Anisotropic strain energy density

$$\Psi_f(I_4, I_6) = \frac{k_1}{2k_2} \sum_{i=4,6} \{ \exp[k_2(I_i - 1)^2] - 1 \},$$

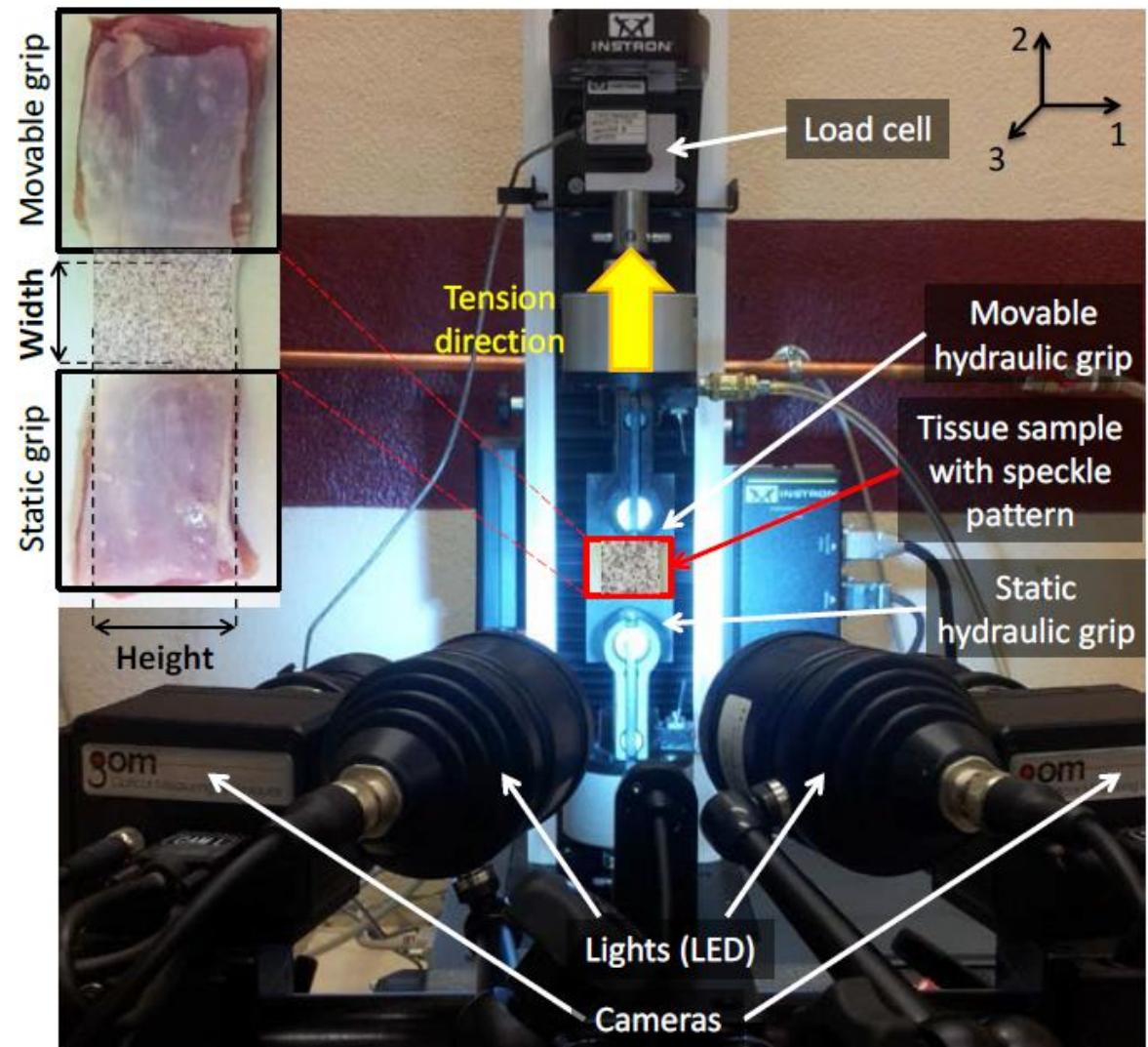
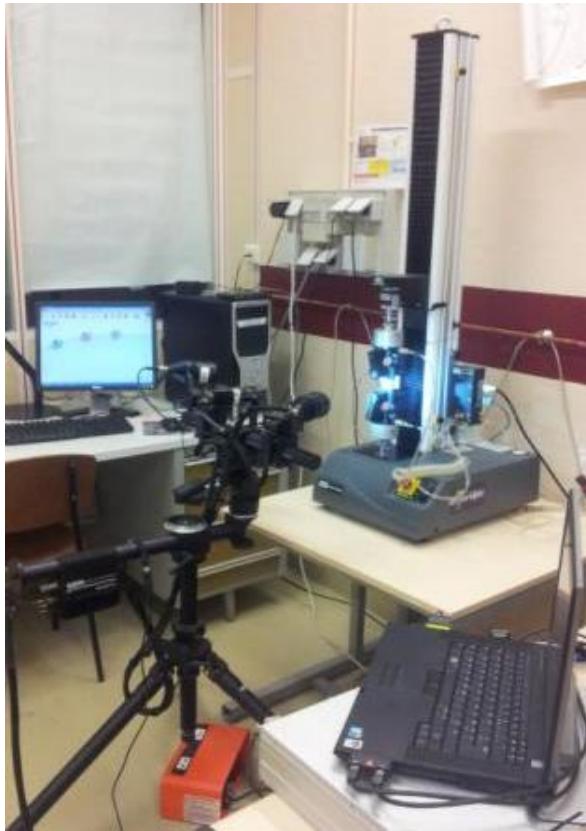
(Holzapfel et al., 2000)





The use of full-field measurements in nonlinear elasticity

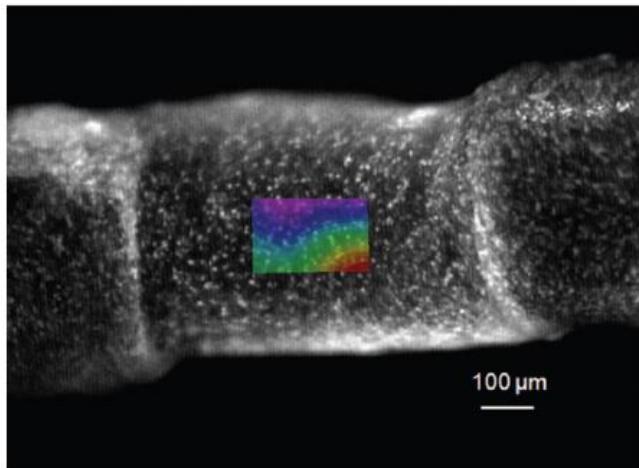
DIC tend to be used for all kind of soft tissues



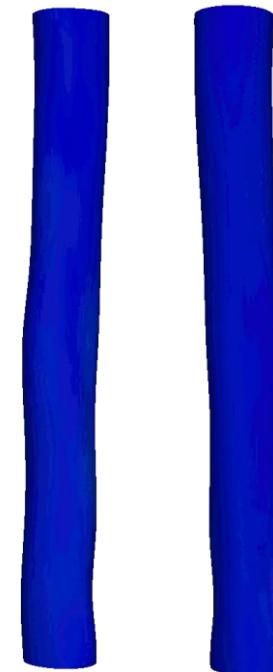
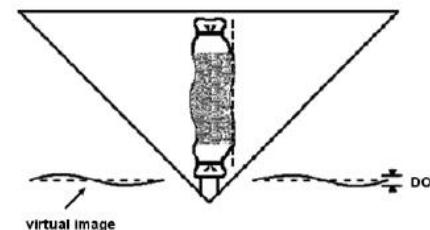
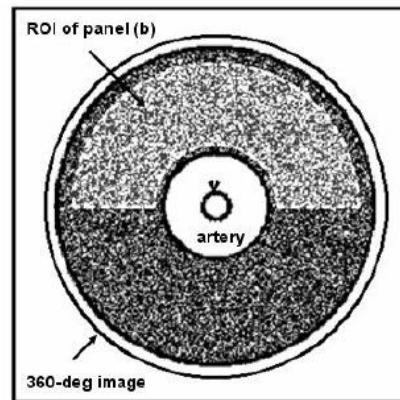
MEASUREMENT OF THE RESPONSE USING DIGITAL IMAGE CORRELATION



classical



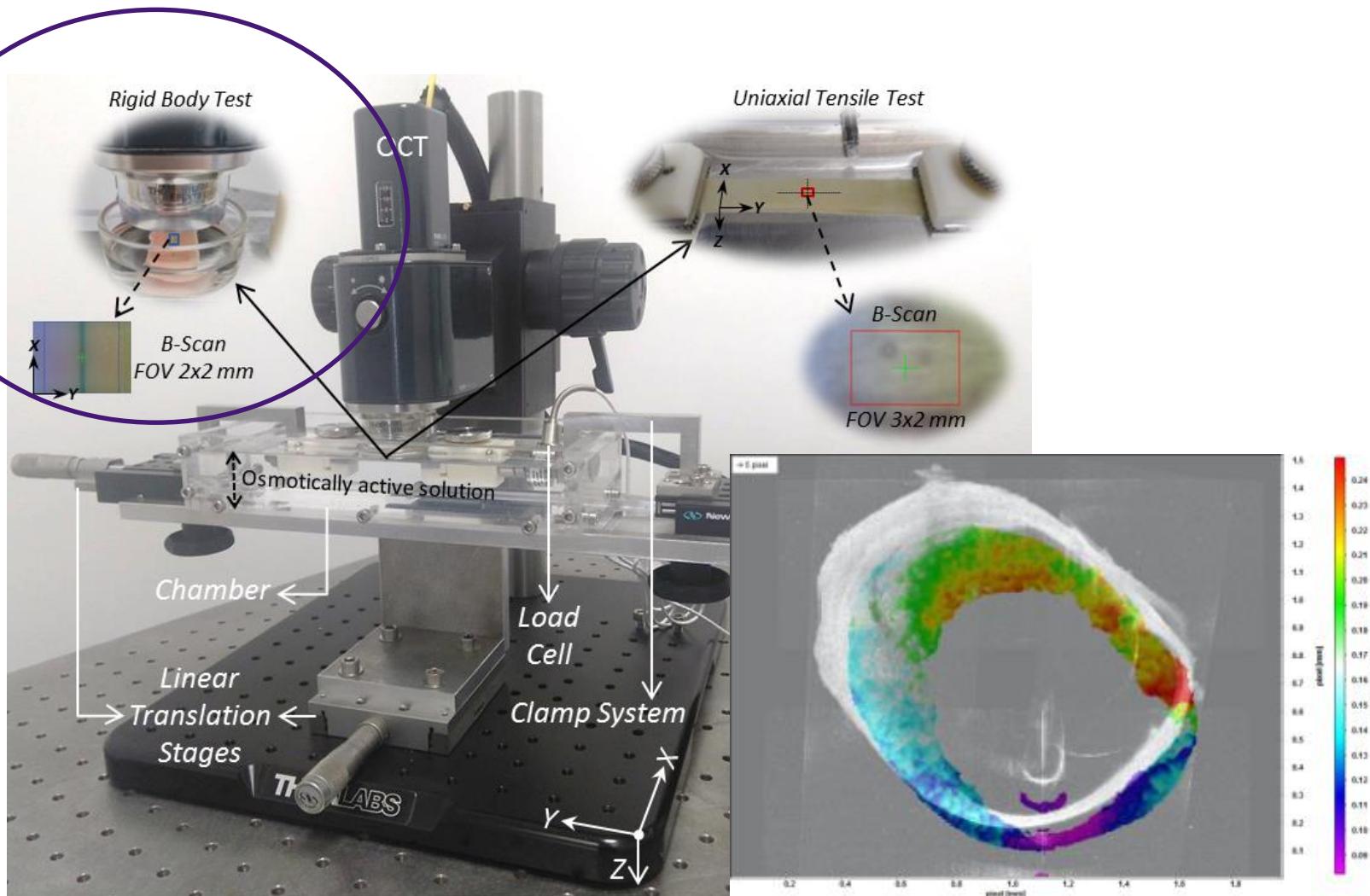
panoramic



Badel et al. CMBBE, **15**, p 37-48, 2012.

Genovese. Optics Lasers Eng, **47**, p 995-1008, 2009.

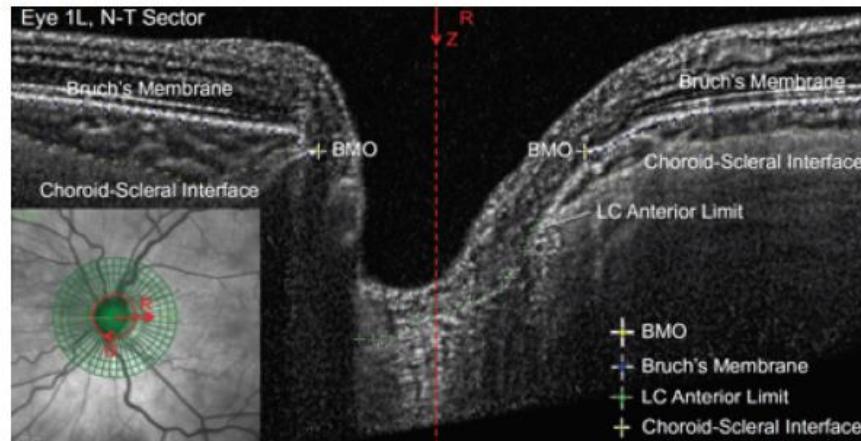
OCT-DVC applied to soft tissue mechanics



Specific problem of the optical nerve head

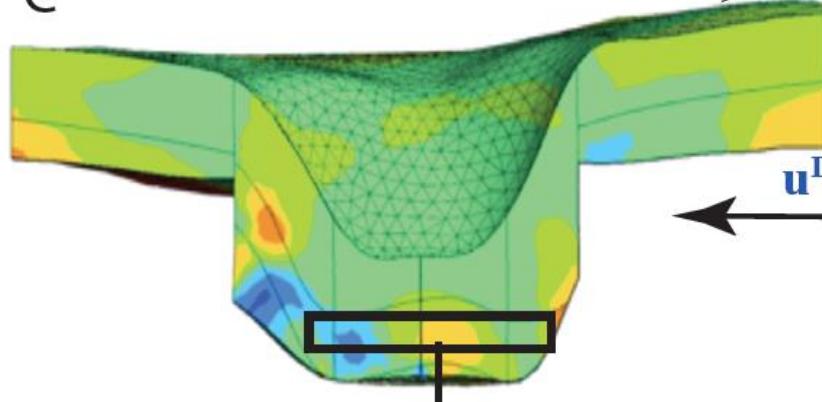


A



in vivo OCT

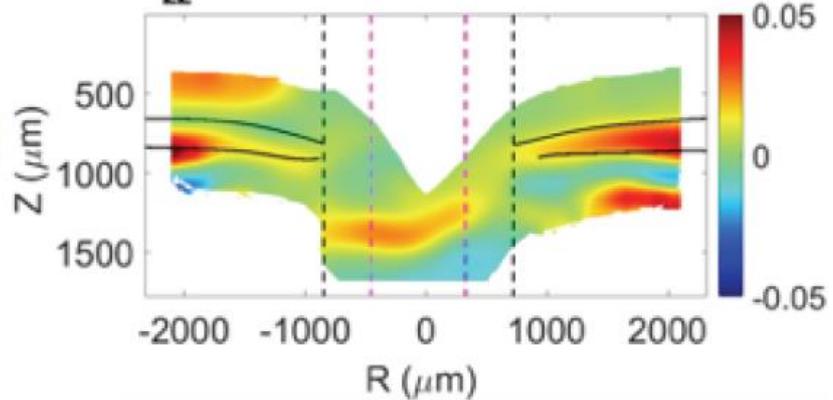
C



DVC

B

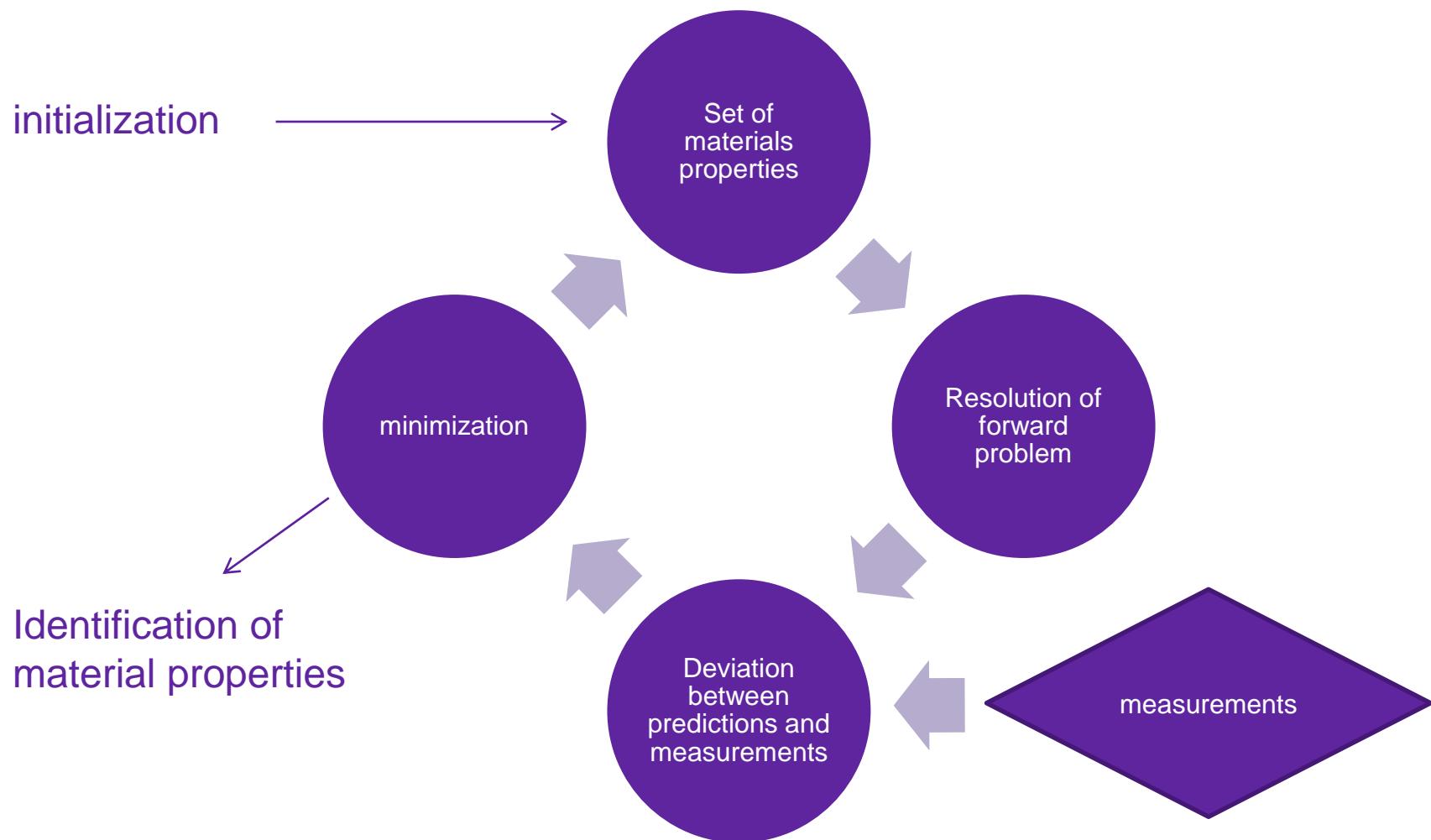
E_{zz} Strain, Nasal-Temporal Sector



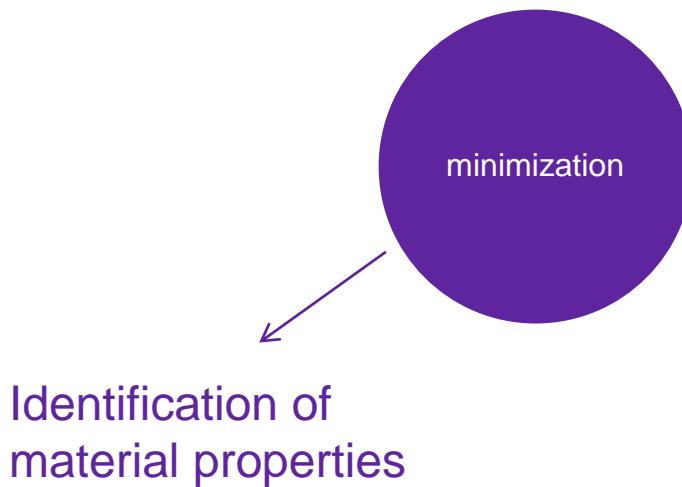


Identification of hyperelastic parameters: state of the art

Inverse approach – traditional FEMU approach



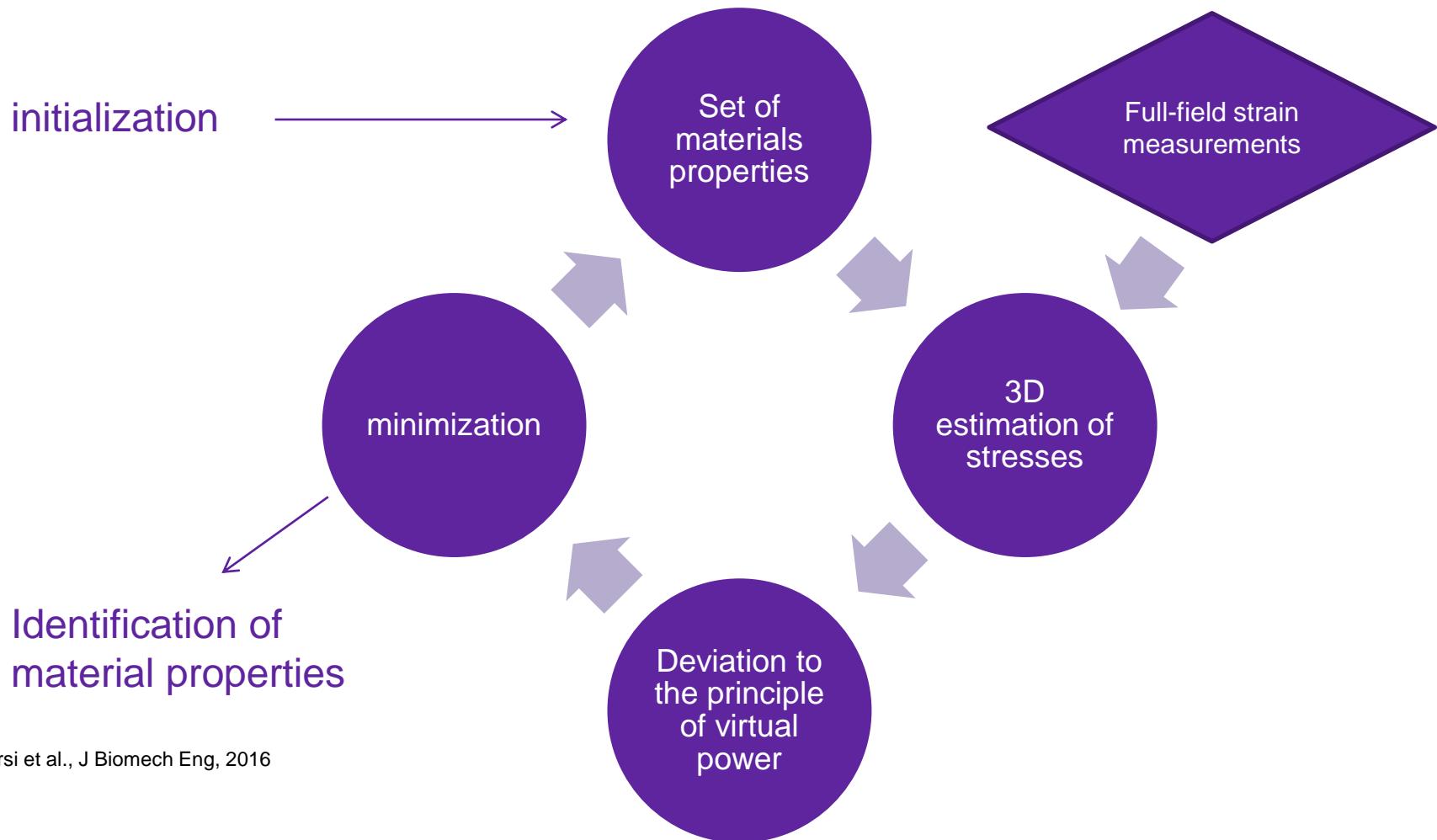
FEMU approach - limitations



1. Use a gradient based method (steepest descent or BFGS)
2. Need to derive the gradient of J with respect to μ at each iteration. Many iterations
3. Very unstable with hyperelastic models: **many risks that the forward problems have a poor convergence**



The virtual fields method (VFM)



Bersi et al., J Biomech Eng, 2016

The principle of virtual work

In statics, equilibrium may be written as follows :

$$\sigma_{jj,j} + f_j = 0 \text{ + boundary conditions} \quad \textit{strong form}$$

or

$$-\int_V \sigma_{jj} \varepsilon_{jj}^* dV + \int_{\partial V} T_j u_j^* dS + \int_V f_j u_j^* dV = 0 \quad \textit{weak form}$$

Satisfied for any kinematically admissible (KA)
virtual field u^*



Application to the identification of material parameters: the virtual fields method

Example in incompressible hyperelasticity

$$\underline{\underline{\sigma}} = \rho \underline{\underline{F}} \cdot \frac{\partial \Psi}{\partial \underline{\underline{E}}} \cdot {}^t \underline{\underline{F}} + c \underline{\underline{I}}$$

$$-\int_V \left(\rho \underline{\underline{F}} \cdot \frac{\partial \Psi}{\partial \underline{\underline{E}}} \cdot {}^t \underline{\underline{F}} + c \underline{\underline{I}} \right) : \underline{\underline{\varepsilon}}^* dV + \int_{\partial V} \underline{T} \cdot \underline{u}^* dS = 0$$

Application to the identification of material parameters: the virtual fields method

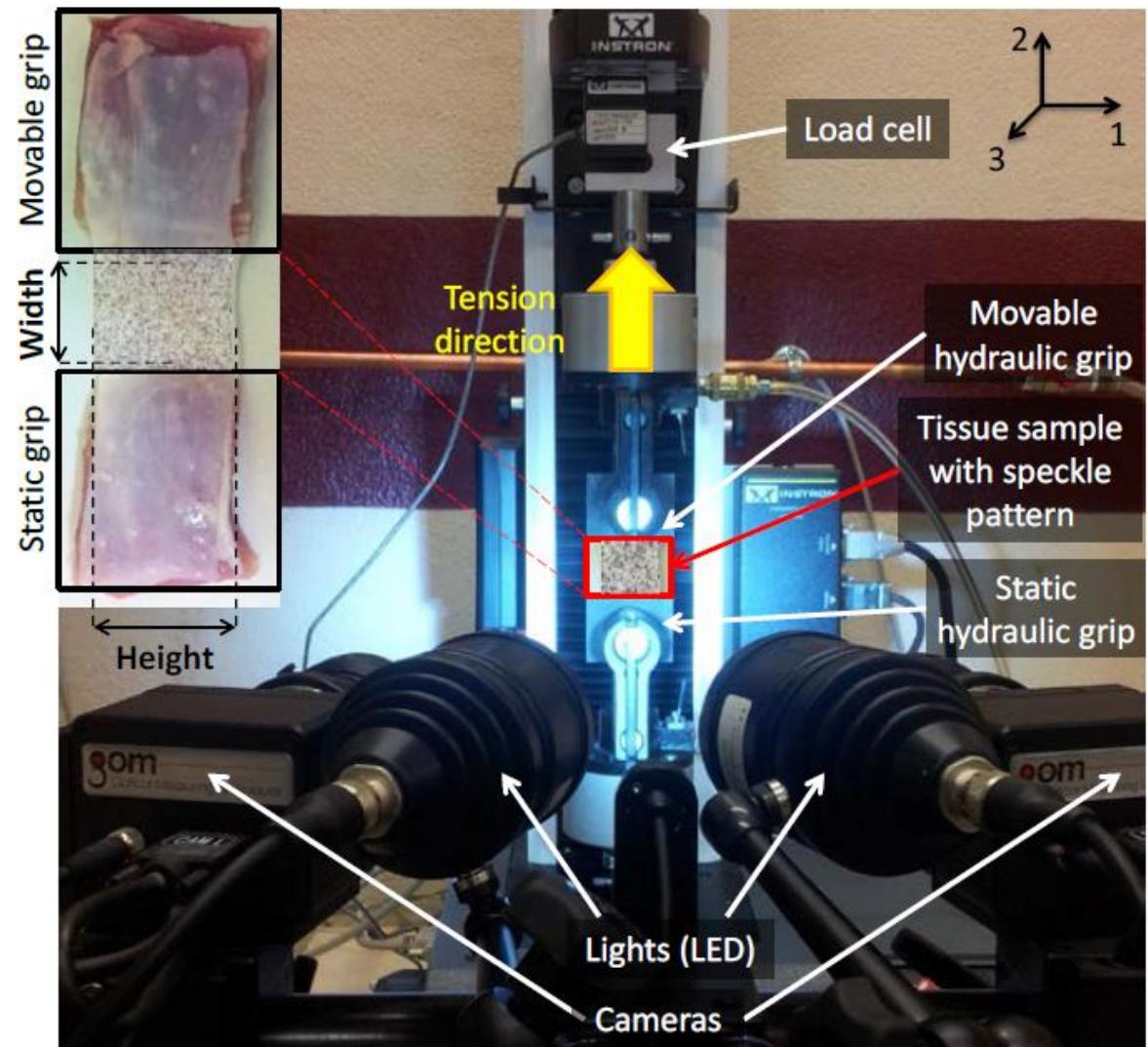
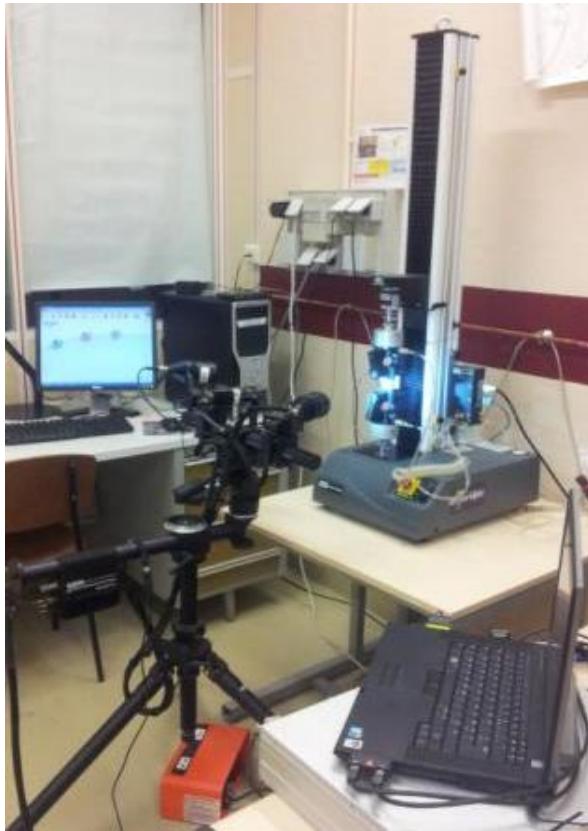
Example for NeoHooke incompressible

$$\underline{\underline{\sigma}} = 2C_{10}\underline{\underline{F}}^t \underline{\underline{F}} + c\underline{\underline{I}}$$

$$2C_{10} \int_V (\underline{\underline{F}}^t \underline{\underline{F}} + c\underline{\underline{I}}) : \underline{\underline{\varepsilon}}^* dV = \int_{\partial V} \underline{\underline{T}} \cdot \underline{\underline{u}}^* dS$$

$$C_{10} = \frac{\int_{\partial V} \underline{\underline{T}} \cdot \underline{\underline{u}}^* dS}{2 \int_V (\underline{\underline{F}}^t \underline{\underline{F}} + c\underline{\underline{I}}) : \underline{\underline{\varepsilon}}^* dV} = \frac{\int_{\partial V} \underline{\underline{T}} \cdot \underline{\underline{u}}^* dS}{2 \int_V (\underline{\underline{B}} + c\underline{\underline{I}}) : \underline{\underline{\varepsilon}}^* dV}$$

Application in simple uniaxial tension

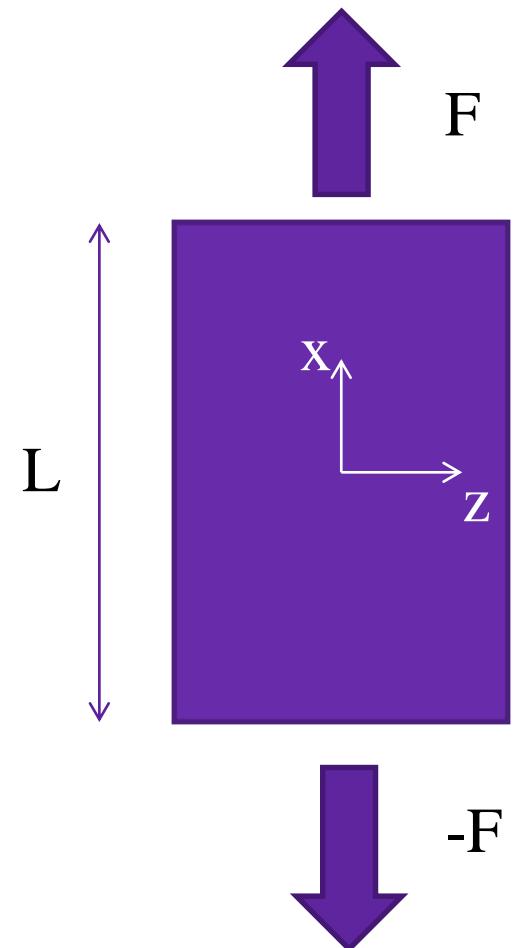


Application in simple uniaxial tension

$$\underline{u}^* = \begin{Bmatrix} x \\ -y/2 \\ -z/2 \end{Bmatrix}$$

$$\int \frac{\underline{T} \cdot \underline{u}^*}{\partial V} dS = FL$$

$$C_{10} = \frac{FL}{\int_V (B_{11} - B_{22} - B_{33}) dV}$$



VFM approach - limitations

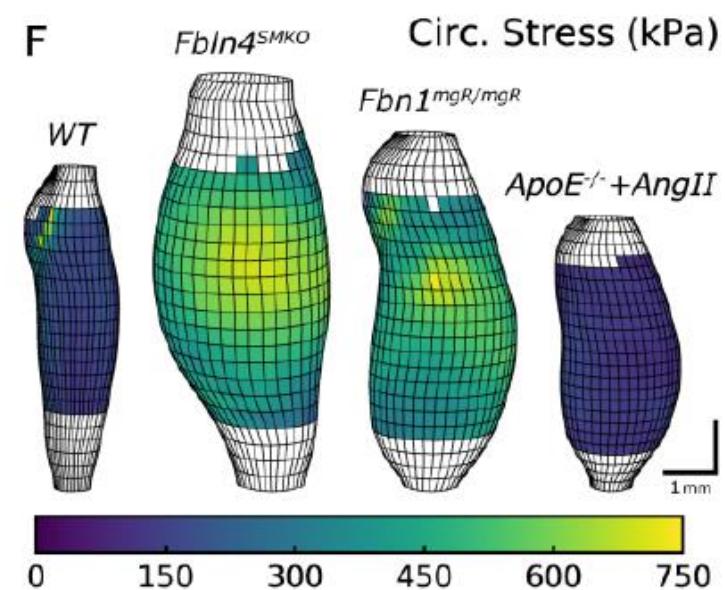
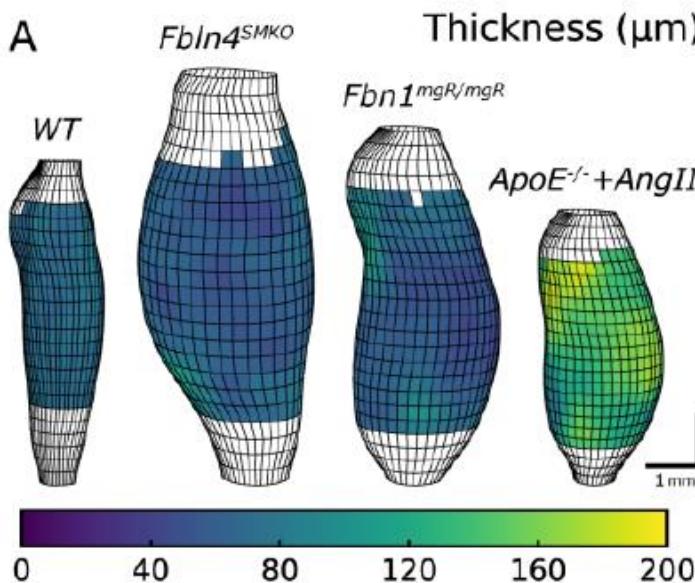
Customized
cost function

Deviation to
the principle
of virtual
power

1. Method that can provide virtual fields for any 3D geometry
2. independent virtual fields are needed in order to separate the hydrostatic and deviatoric contributions of the stress for compressible hyperelastic materials
3. **Lack of automatic approach**

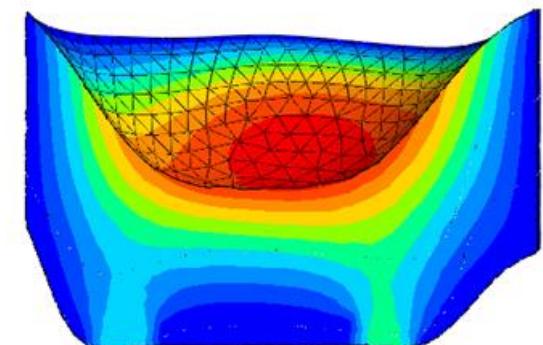
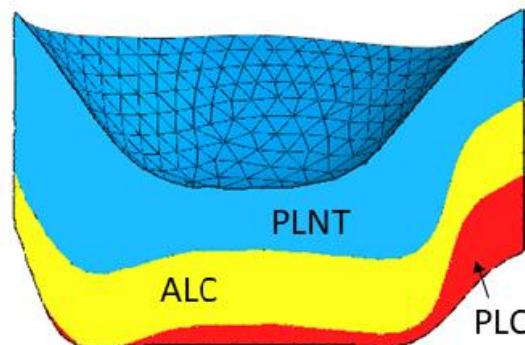
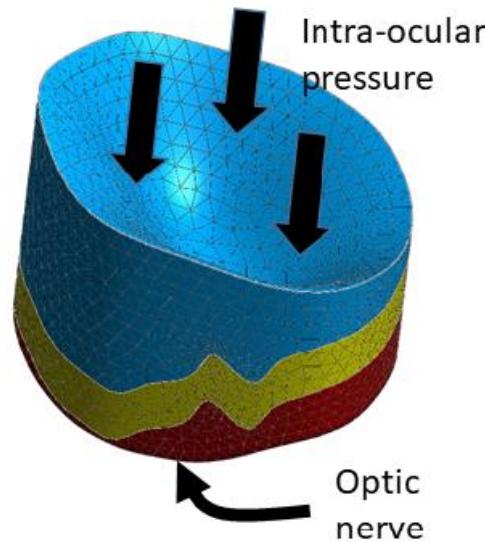


Successful applications for reconstructing local hyperelastic properties of arteries took 3 years...



Bersi et al, BMMB 2018, Bersi et al, Scientific Reports 2020

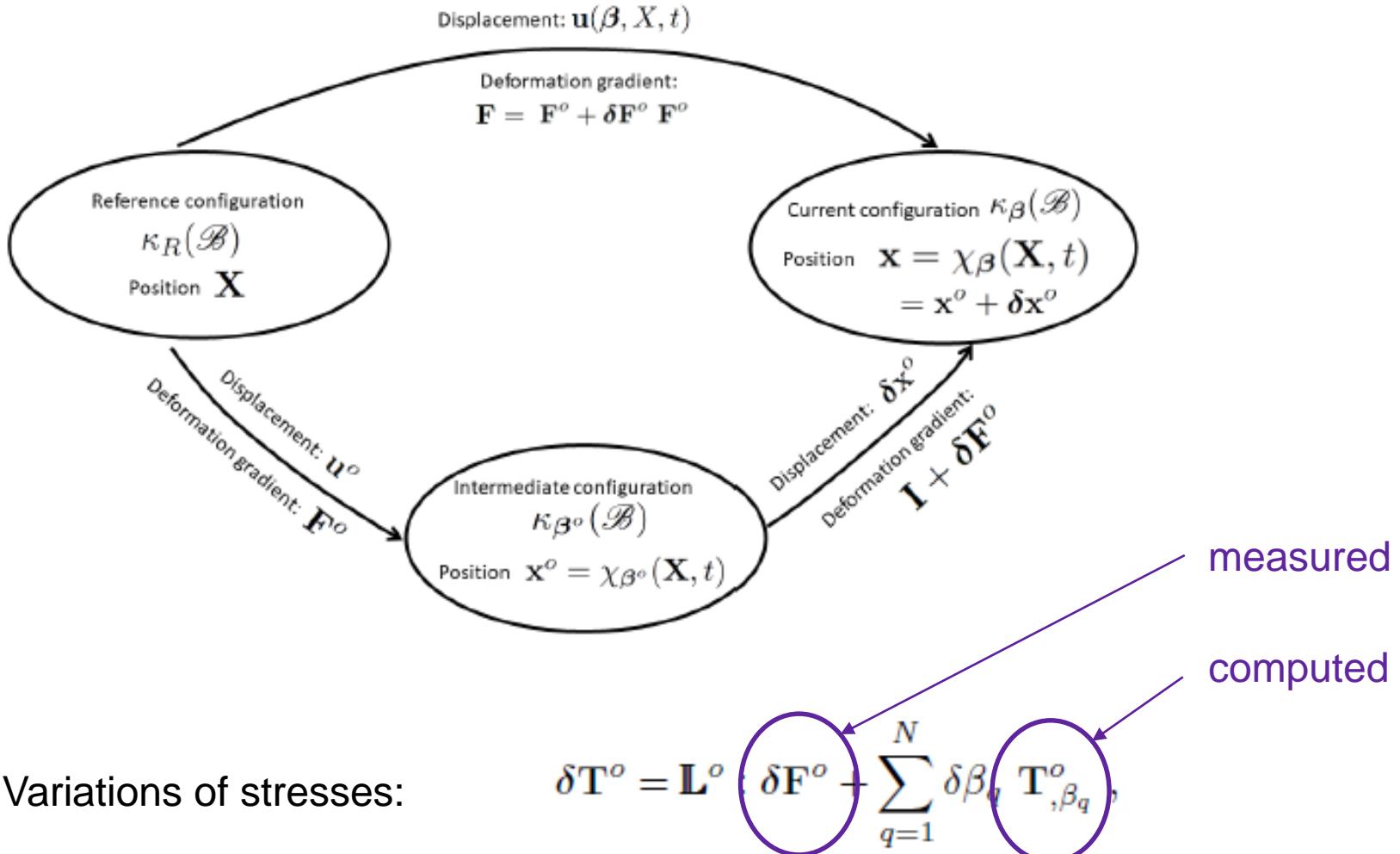
How to apply the VFM on the optical nerve head problem without the lengthy trial and error stage?





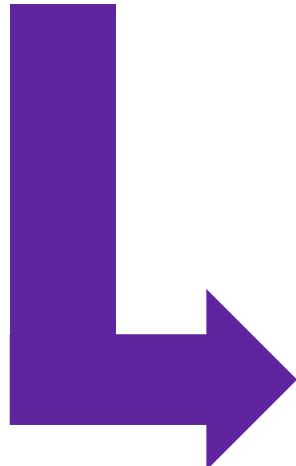
Identification of hyperelastic parameters: our new VFM approach

Varying the material properties result in variations of stresses



Variations of stresses should satisfy the principle of virtual power

$$\int_{\chi_{\beta^o}(\mathcal{B},t)} \delta T^o : \nabla \delta u^{o(n)} dV^o = 0 ,$$



$$\begin{aligned}
 & \left[\begin{array}{c} \int_{\chi_{\beta^o}(\mathcal{B})} T_{,\beta_1}^o : \nabla \delta u^{o(1)} dV^o \\ \vdots \\ \int_{\chi_{\beta^o}(\mathcal{B})} T_{,\beta_N}^o : \nabla \delta u^{o(1)} dV^o \end{array} \dots \begin{array}{c} \int_{\chi_{\beta^o}(\mathcal{B})} T_{,\beta_1}^o : \nabla \delta u^{o(N)} dV^o \\ \vdots \\ \int_{\chi_{\beta^o}(\mathcal{B})} T_{,\beta_N}^o : \nabla \delta u^{o(N)} dV^o \end{array} \right] \\
 & \times \begin{pmatrix} \delta \beta_1^o \\ \vdots \\ \delta \beta_N^o \end{pmatrix} = - \begin{pmatrix} \int_{\chi_{\beta^o}(\mathcal{B})} \tilde{H} : (\mathbf{L}^{oT} : \nabla \delta u^{o(1)}) dV^o \\ \text{measured} \\ \vdots \\ \int_{\chi_{\beta^o}(\mathcal{B})} \tilde{H} : (\mathbf{L}^{oT} : \nabla \delta u^{o(N)}) dV^o \end{pmatrix} \text{,} \quad ?
 \end{aligned}$$

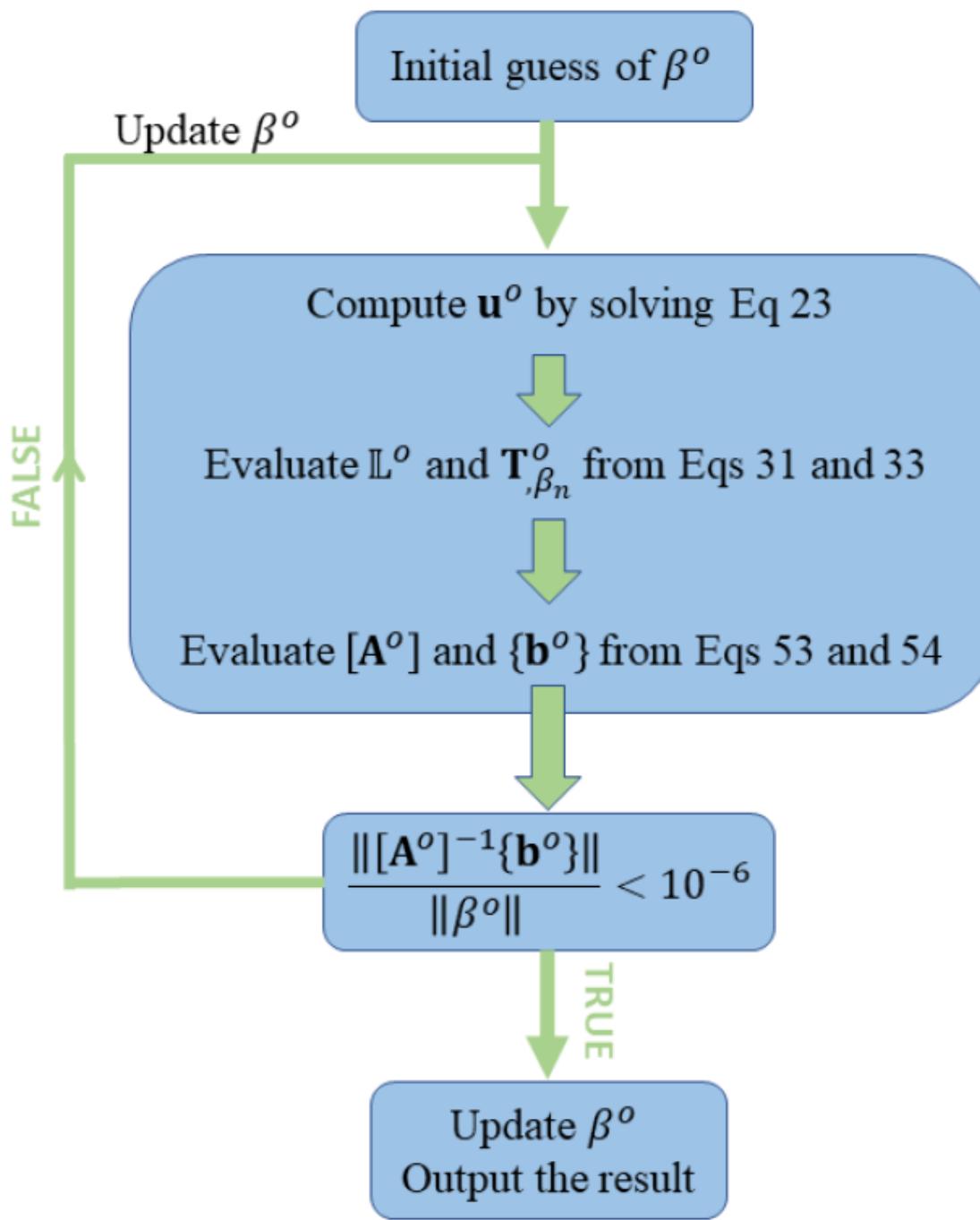
Variations of stresses should satisfy the principle of virtual power

$$\begin{bmatrix}
 \int_{\chi_{\beta^o}(\mathcal{B})} \mathbf{T}_{,\beta_1}^o : \nabla \mathfrak{L}(\mathbf{T}_{,\beta_1}^o) dV^o & \dots & \int_{\chi_{\beta^o}(\mathcal{B})} \mathbf{T}_{,\beta_N}^o : \nabla \mathfrak{L}(\mathbf{T}_{,\beta_1}^o) dV^o \\
 \vdots & \ddots & \vdots \\
 \int_{\chi_{\beta^o}(\mathcal{B})} \mathbf{T}_{,\beta_1}^o : \nabla \mathfrak{L}(\mathbf{T}_{,\beta_N}^o) dV^o & \dots & \int_{\chi_{\beta^o}(\mathcal{B})} \mathbf{T}_{,\beta_N}^o : \nabla \mathfrak{L}(\mathbf{T}_{,\beta_N}^o) dV^o
 \end{bmatrix} \times \begin{pmatrix} \delta \beta_1^o \\ \delta \beta_2^o \\ \vdots \\ \delta \beta_N^o \end{pmatrix} = - \begin{pmatrix} \int_{\chi_{\beta^o}(\mathcal{B})} \tilde{\mathbf{H}} : \mathbf{T}_{,\beta_1}^o dV^o \\ \vdots \\ \int_{\chi_{\beta^o}(\mathcal{B})} \tilde{\mathbf{H}} : \mathbf{T}_{,\beta_N}^o dV^o \end{pmatrix}. \quad (44)$$

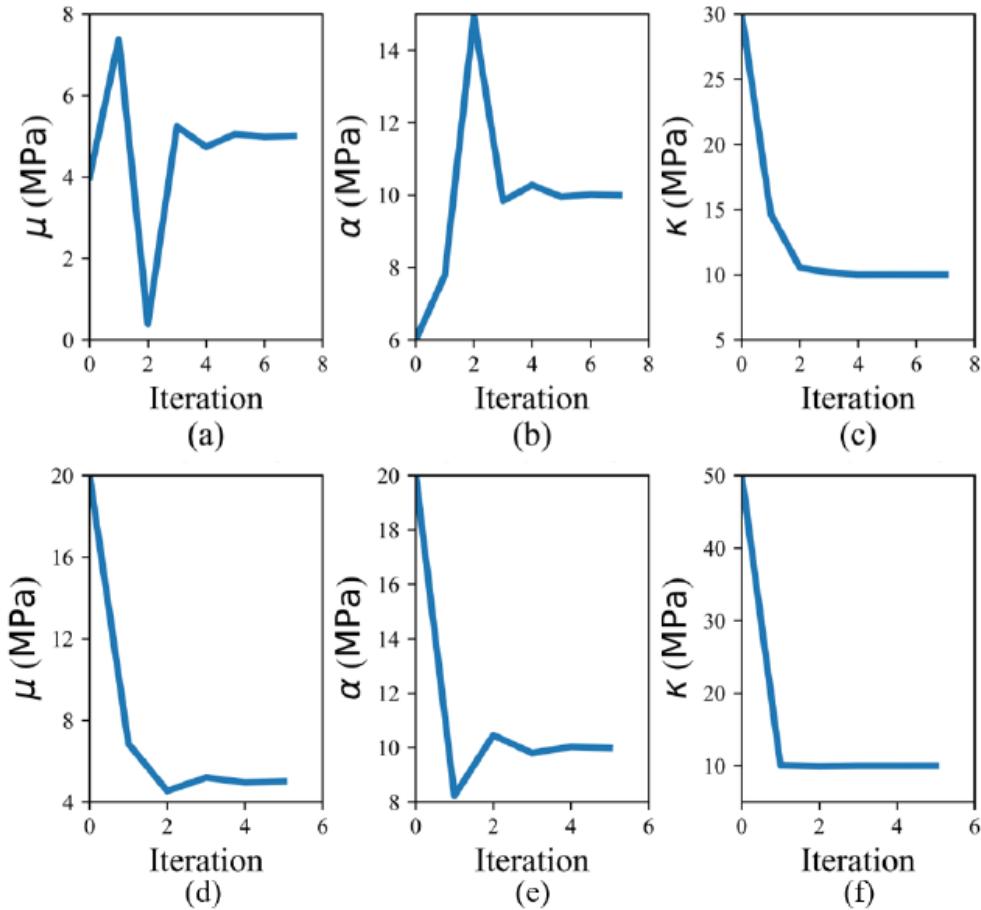
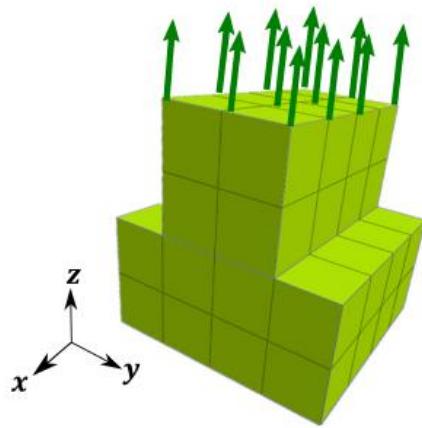
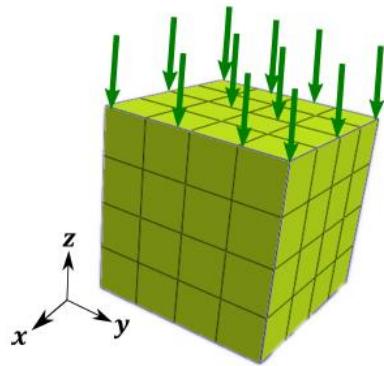
computed

measured

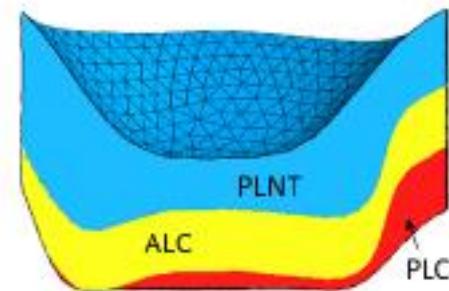
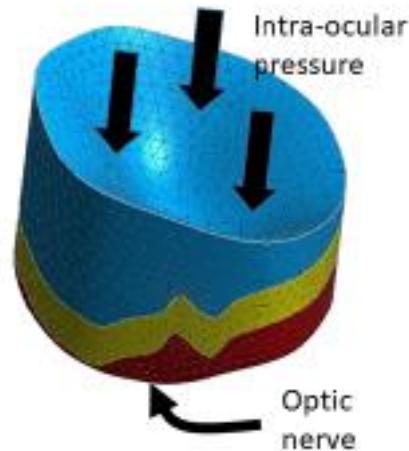
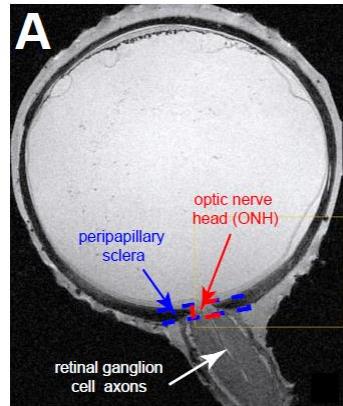
computed



Quadratic convergence

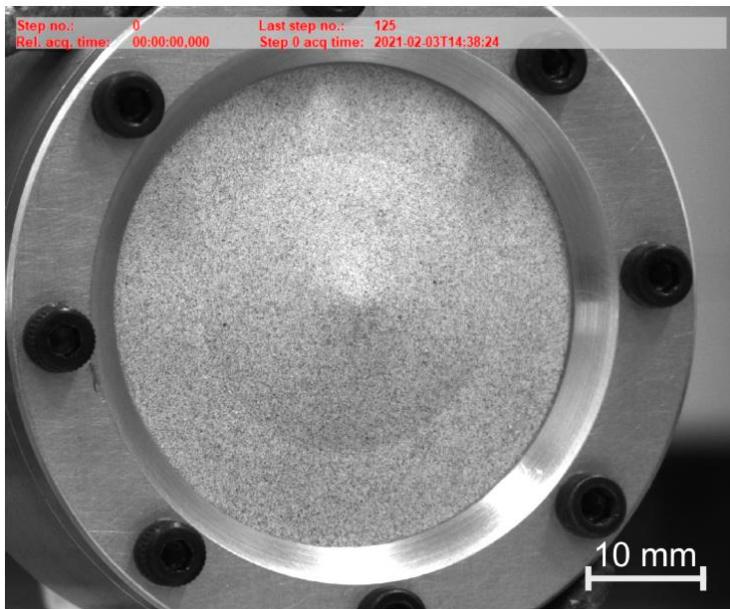


Application to the eye problem

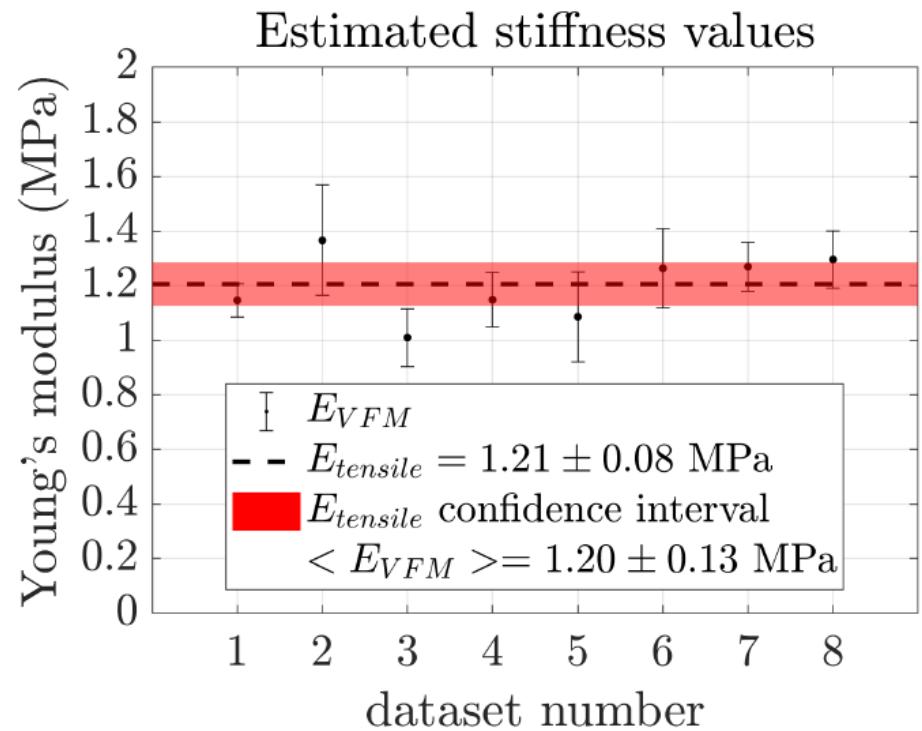


	Reference κ	Identified κ after 4 iterations	relative error	Identified κ after 5 iterations	relative error
PLNT	2.29 MPa	2.31 MPa	0.82%	2.284 MPa	0.35%
ALC	2.5 MPa	2.51 MPa	0.55%	2.495 MPa	0.22%
PLC	2.71 MPa	2.72 MPa	0.37%	2.706 MPa	0.08%

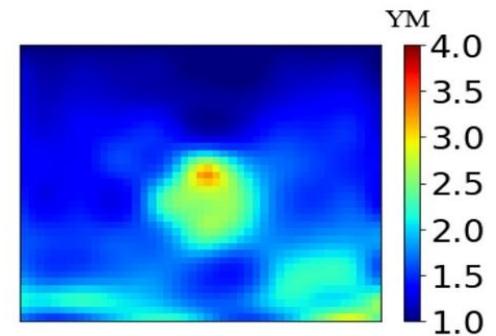
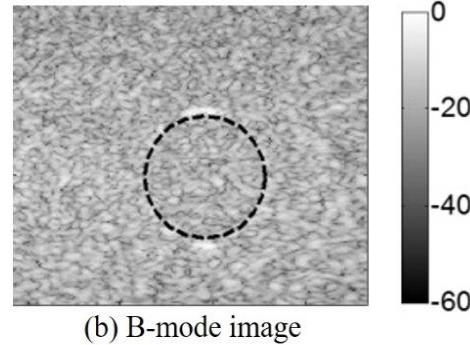
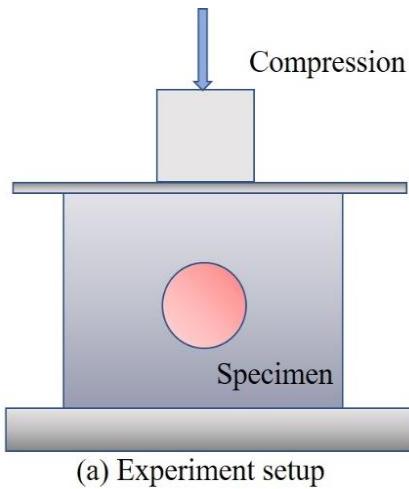
Applications to other problems



Rubber membrane



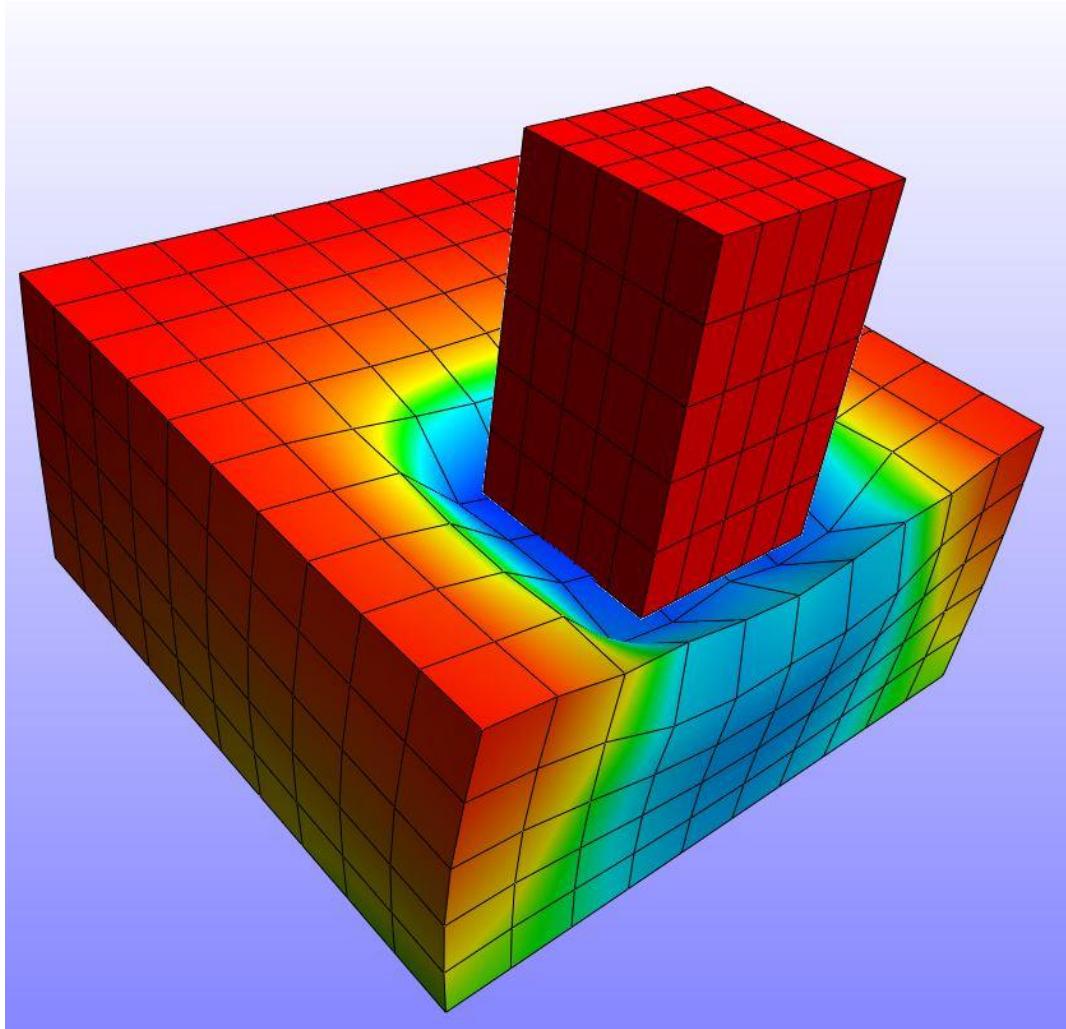
Applications to other problems



Elastography



Towards finite element software for identification of material parameters in nonlinear elasticity



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