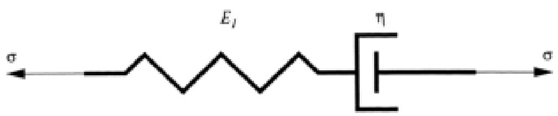
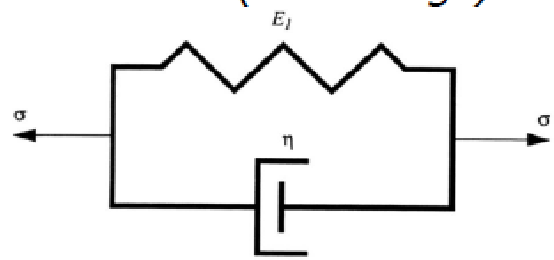
Simple Viscoelastic Models

Stress depends on strain *and* strain-rate:

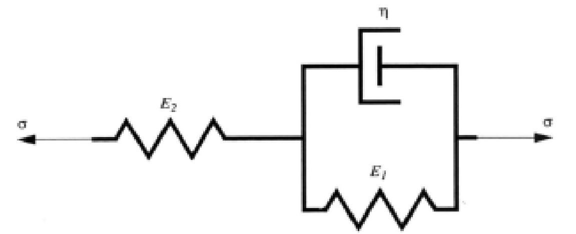
Maxwell



Kelvin *(aka Voigt)*



Standard Linear Solid



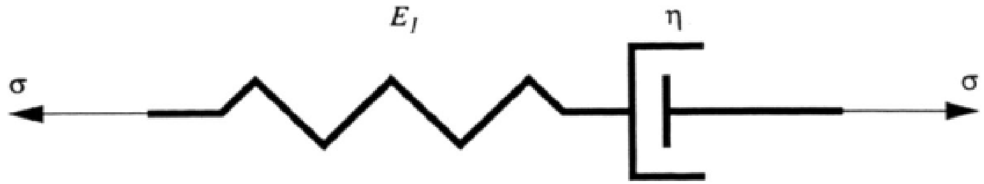
- Elastic stress depends on strain (spring)
- Viscous stress depends on strain-rate (damper)

- Maxwell: Strains add in series, stresses are equal

- Kelvin: Stresses add in parallel, strains are equal

- SLS: Combination of Maxwell and Kelvin

Maxwell Model



Total strain = spring strain + dashpot strain:

$$\varepsilon = \varepsilon_{E_1} + \varepsilon_\eta \quad \Rightarrow \dot{\varepsilon} = \dot{\varepsilon}_{E_1} + \dot{\varepsilon}_\eta$$

$$\sigma = \sigma_{E_1} = \sigma_\eta \qquad \sigma_{E_1} = E_1 \cdot \varepsilon_{E_1} \qquad \sigma_\eta = \eta \cdot \dot{\varepsilon}_\eta$$

$$\Rightarrow \dot{\varepsilon}_{E_1} = \frac{\dot{\sigma}}{E_1} \qquad \dot{\varepsilon}_\eta = \frac{\sigma}{\eta}$$

$$\boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}}$$

A linear first-order ordinary differential equation (ODE)

Creep Solution

Does the model creep?

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}$$

$$\frac{d\varepsilon}{dt} = \frac{1}{E_1}\frac{d\sigma}{dt} + \frac{\sigma}{\eta}$$

$$d\varepsilon = \frac{1}{E_1}d\sigma + \frac{\sigma}{\eta}dt$$

Integrating, for constant applied stress, σ_0 :

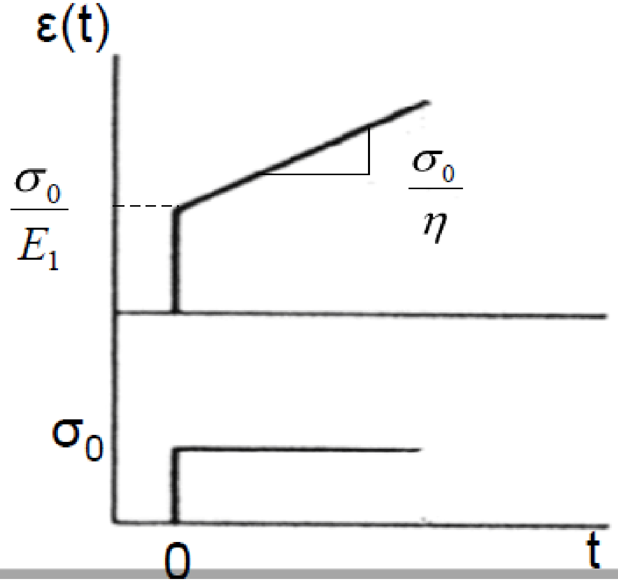
$$\int_{\varepsilon(0)}^{\varepsilon(t)} d\varepsilon = \frac{1}{E_1}\int_{\sigma(0)}^{\sigma(t)} d\sigma + \frac{\sigma_0}{\eta}\int_0^t dt + C \Rightarrow \quad \varepsilon(t) = \varepsilon(0) + \frac{\sigma_0 t}{\eta} + C$$

Creep Solution

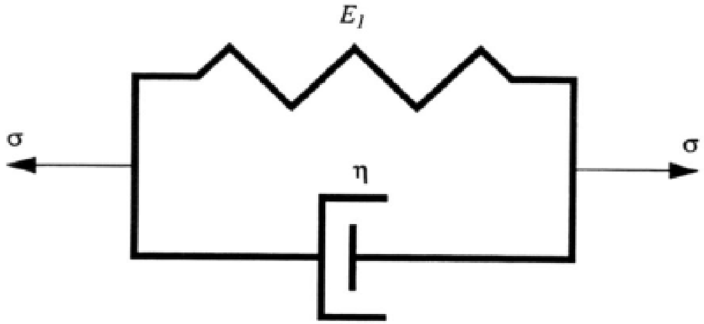
$$\varepsilon(t) = \varepsilon(0) + \frac{\sigma_0 t}{\eta} + C$$

Only the Hooke element reacts initially: $\varepsilon(0) = C = \dfrac{\sigma_0}{E_1}$

$$\Rightarrow \varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0 t}{\eta} = \sigma_0\left(\frac{1}{E_1} + \frac{t}{\eta}\right) = \sigma_0 \underbrace{J(t)}_{\text{Creep function}}$$



Kelvin Model



Total stress = spring stress + dashpot stress:

$$\sigma = \sigma_{E_1} + \sigma_{\eta} \qquad \varepsilon = \varepsilon_{E_1} = \varepsilon_{\eta}$$

$$\sigma_{E_1} = E_1 \cdot \varepsilon \qquad \sigma_{\eta} = \eta \cdot \dot{\varepsilon}$$

$$\boxed{\sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon}}$$

A linear first-order ordinary differential equation (ODE)

Creep Solution

$$\sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon}$$

Does the model creep?

Constant stress, σ_0 : $\dfrac{d\varepsilon}{dt} + \dfrac{E_1}{\eta}\varepsilon = \dfrac{\sigma_0}{\eta}$ $\qquad \varepsilon(t) = \varepsilon_H(t) + \varepsilon_N(t)$

$$\frac{d\varepsilon_H}{dt} + \frac{E_1}{\eta}\varepsilon_H = 0 \Rightarrow \frac{d\varepsilon_H}{\varepsilon_H} = -\frac{E_1}{\eta}dt \Rightarrow \varepsilon_H(t) = C_1 e^{-\frac{E_1}{\eta}t}$$

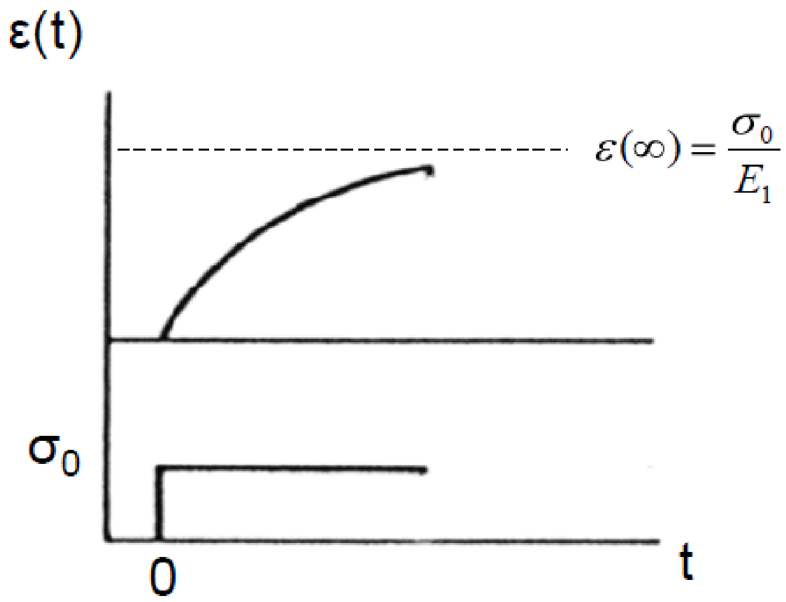
$$\varepsilon_N(t) = C_2 \Rightarrow \varepsilon(t) = C_1 e^{-\frac{E_1}{\eta}t} + C_2 \Rightarrow \frac{d\varepsilon(t)}{dt} = -\frac{E_1}{\eta}C_1 e^{-\frac{E_1}{\eta}t}$$

$$-\frac{E_1}{\eta}C_1 e^{-\frac{E_1}{\eta}t} + \frac{E_1}{\eta}C_1 e^{-\frac{E_1}{\eta}t} + \frac{E_1}{\eta}C_2 = \frac{\sigma_0}{\eta} \Rightarrow C_2 = \frac{\sigma_0}{E_1}$$

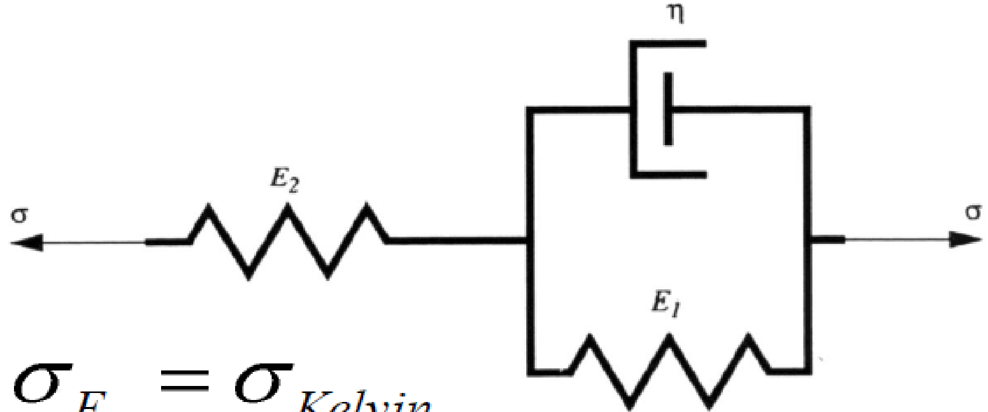
$$\Rightarrow \varepsilon(t) = C_1 e^{-\frac{E_1}{\eta}t} + \frac{\sigma_0}{E_1}$$

Creep Solution

$$\varepsilon(0)=0 \Rightarrow C_1=-\frac{\sigma_0}{E_1} \qquad \Rightarrow \varepsilon(t)=\frac{\sigma_0}{E_1}(1-e^{-\frac{E_1}{\eta}t})=\sigma_0 J(t)$$



Standard Linear Solid



$$\sigma = \sigma_{E_2} = \sigma_{Kelvin}$$

$$\varepsilon = \varepsilon_{E_2} + \varepsilon_{Kelvin} \qquad \dot{\varepsilon} = \dot{\varepsilon}_{E_2} + \dot{\varepsilon}_{Kelvin}$$

$$\varepsilon_{E_2} = \frac{\sigma}{E_2} \Rightarrow \dot{\varepsilon}_{E_2} = \frac{\dot{\sigma}}{E_2}$$

$$\sigma_{Kelvin} = E_1 \cdot \varepsilon_{Kelvin} + \eta \cdot \dot{\varepsilon}_{Kelvin} \Rightarrow \dot{\varepsilon}_{Kelvin} = (\sigma - E_1 \cdot \varepsilon_{Kelvin}) / \eta$$

Standard Linear Solid

$$\Rightarrow \dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + (\sigma - E_1 \cdot \varepsilon_{kelvin}) / \eta$$

$$\varepsilon_{kelvin} = \varepsilon - \varepsilon_{E_2} \Rightarrow$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta} - \frac{E_1}{\eta} \cdot (\varepsilon - \varepsilon_{E_2}) \Rightarrow$$

$$\varepsilon_{E_2} = \frac{\sigma}{E_2} \Rightarrow$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta}(1 + \frac{E_1}{E_2}) - \frac{E_1}{\eta}\varepsilon$$

Creep Solution

Does the model creep?

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta}\left(1 + \frac{E_1}{E_2}\right) - \frac{E_1}{\eta}\varepsilon$$

A constant stress, σ_0 is instantaneously applied at time t=0,
Constant stress ➔ $d\sigma/dt=0$, when $t>0$

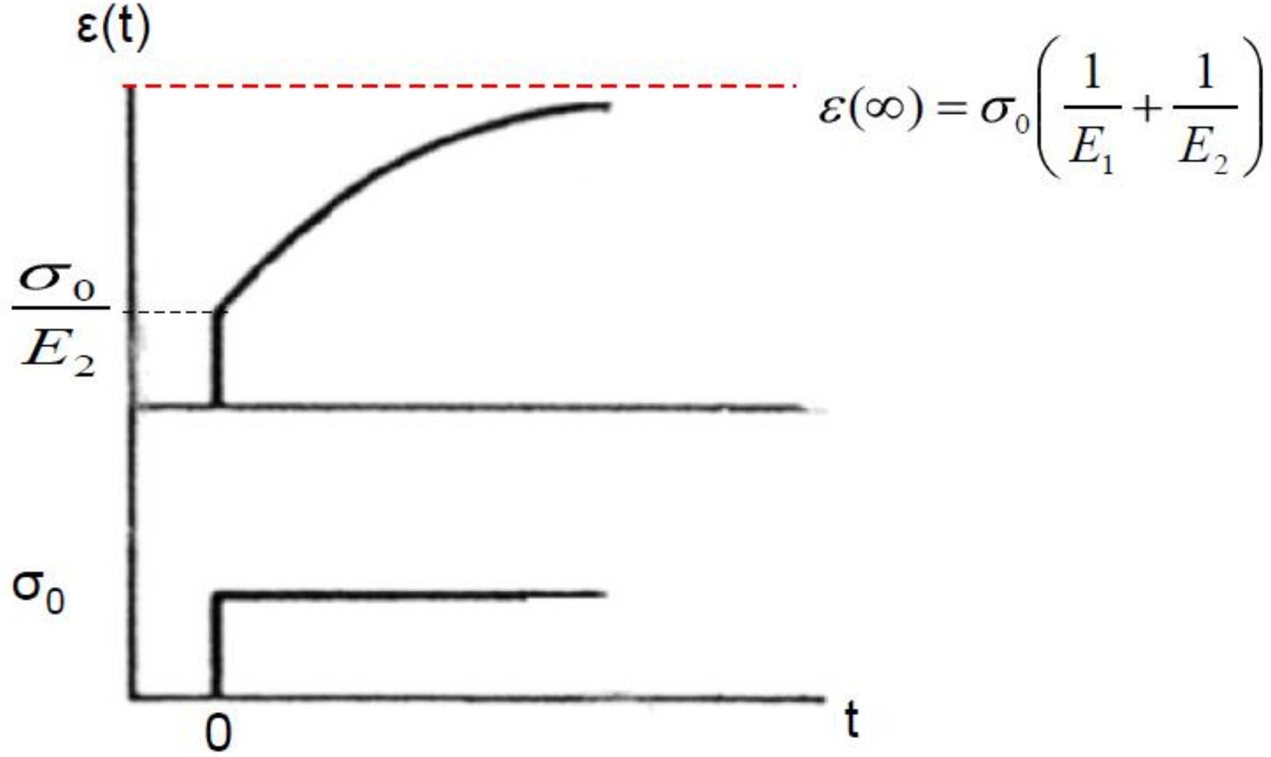
$$\frac{d\varepsilon}{dt} + \frac{E_1}{\eta}\varepsilon = \frac{\sigma_0}{\eta}\left(1 + \frac{E_1}{E_2}\right) \Rightarrow$$

A linear first-order ordinary differential equation (ODE)

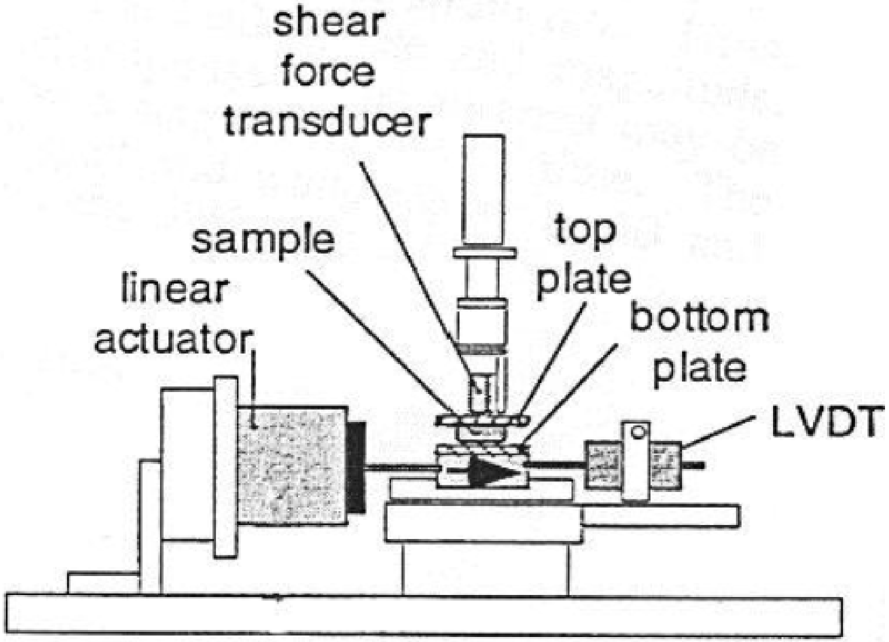
Solving, and using the fact that only the Hooke element reacts initially $\Rightarrow \varepsilon(0) = \dfrac{\sigma_0}{E_2} \Rightarrow$

Creep Solution

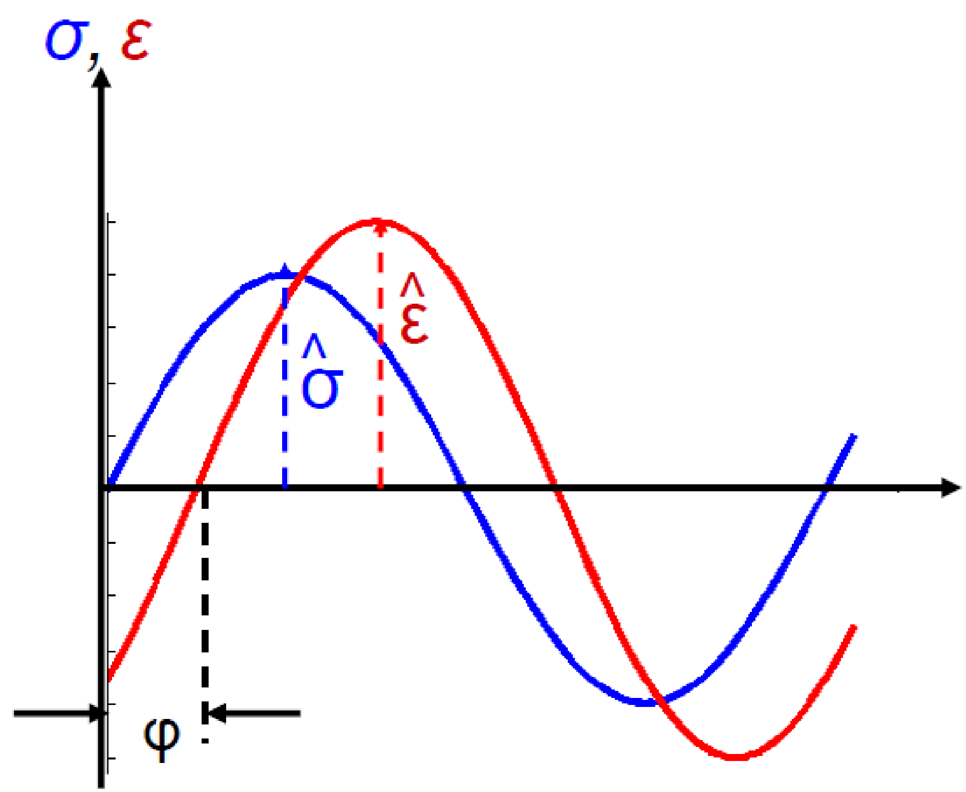
$$\ldots \Rightarrow \varepsilon(t) = \sigma_0 \left[\frac{1}{E_1}\left(1 - e^{-\frac{E_1}{\eta}t}\right) + \frac{1}{E_2} \right] = \sigma_0 J(t)$$



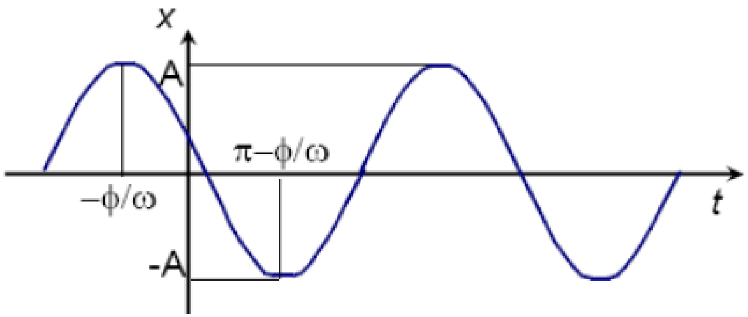
Oscillations to determine viscoelastic properties

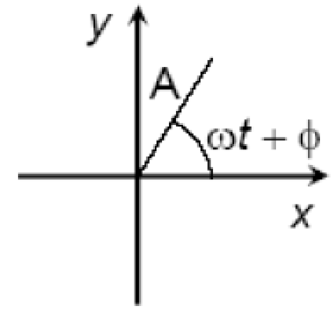


$$\varepsilon = \hat{\varepsilon} \sin(\omega t)$$



Harmonic Loading





$$y = A\sin(\omega t + \phi)$$
$$x = A\cos(\omega t + \phi)$$

$$z = x + iy = A\left[\cos(\omega t + \phi) + i\sin(\omega t + \phi)\right]$$
$$= Ae^{i(\omega t + \phi)} \qquad = \underbrace{Ae^{i\phi}}_{B} e^{i\omega t} \qquad = Be^{i\omega t}$$

$$A = amplitude = |B|$$

$$\phi = phase\ \ angle = \tan^{-1}(\operatorname{Im} B / \operatorname{Re} B)$$

$$\omega = angular\ frequency \quad 2\pi f \quad (rad/s)$$

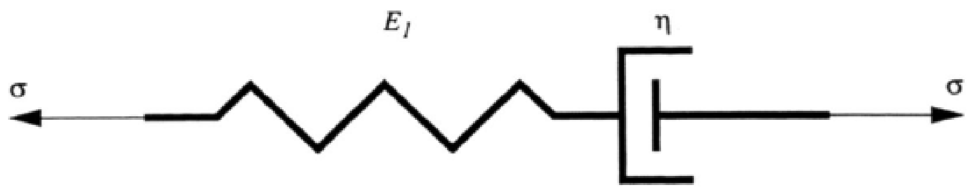
Harmonic Stress and Strain History

$$\varepsilon = \hat{\varepsilon}\cos(\omega t) \qquad \varepsilon = \hat{\varepsilon}\cdot e^{i\omega t}$$

$$\dot{\varepsilon} = i\omega\hat{\varepsilon}\cdot e^{i\omega t} \qquad \Rightarrow \boxed{\dot{\varepsilon} = i\omega\varepsilon}$$

$$\sigma = \hat{\sigma}\cos(\omega t) \quad \ldots \Rightarrow \boxed{\dot{\sigma} = i\omega\sigma}$$

Maxwell Model



$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}$$

$$\Rightarrow i\omega\varepsilon = \sigma\left(\frac{i\omega}{E_1} + \frac{1}{\eta}\right) \qquad \Rightarrow \sigma(i\omega) = \underbrace{i\omega\left(\frac{i\omega}{E_1} + \frac{1}{\eta}\right)^{-1}}_{E(i\omega)}\varepsilon$$

Complex modulus for the Maxwell Model

$$\Rightarrow E(i\omega) = i\omega \frac{1}{\frac{i\omega}{E_1} + \frac{1}{\eta}} \quad \begin{matrix} \times (\frac{i\omega}{E_1} - \frac{1}{\eta}) \\ \times (\frac{i\omega}{E_1} - \frac{1}{\eta}) \end{matrix}$$

$$\Rightarrow E(i\omega) = \frac{-\frac{\omega^2}{E_1} - \frac{i\omega}{\eta}}{-\frac{\omega^2}{E_1{}^2} - \frac{1}{\eta^2}}$$

$$\Rightarrow E(i\omega) = \left(\boxed{\frac{\omega^2}{E_1}} + \boxed{\frac{i\omega}{\eta}}\right) / \left(\frac{\omega^2}{E_1{}^2} + \frac{1}{\eta^2}\right)$$

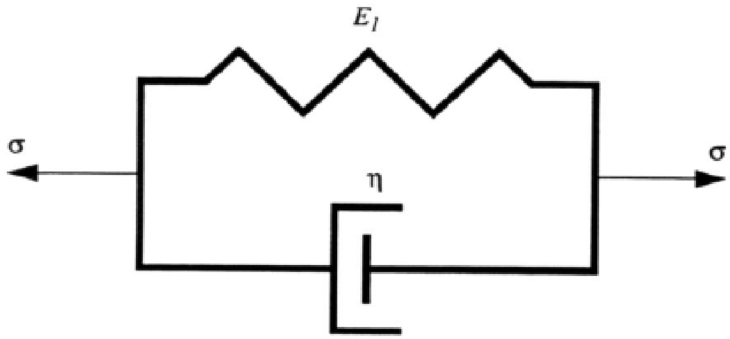
$\boxed{\mathrm{Re}\, E(i\omega)}$ $\boxed{\mathrm{Im}\, E(i\omega)}$

$$E(i\omega) = E' + iE''$$

"Stiffness" *"Damping"*

Complex modulus for the Kelvin Model

Kelvin



$$\sigma = E_1 \cdot \varepsilon + \eta \cdot \dot{\varepsilon}$$

$$\dot{\varepsilon} = i\omega\varepsilon \Rightarrow$$

$$\sigma(i\omega) = \underbrace{(E_1 + i\omega \cdot \eta)}_{E(i\omega)}\varepsilon$$

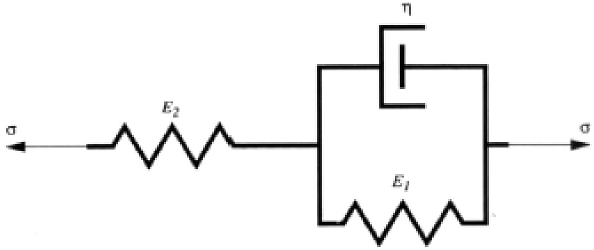
$$E(i\omega) = E_1 + i\omega \cdot \eta$$

$\operatorname{Re} E(i\omega)$ → *"Stiffness"*

$\operatorname{Im} E(i\omega)$ → *"Damping"*

Complex modulus for the SLS Model

Standard Linear Solid



$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta}\left(1 + \frac{E_1}{E_2}\right) - \frac{E_1}{\eta}\varepsilon$$

$$\dot{\varepsilon} = i\omega\varepsilon$$

$$\dot{\sigma} = i\omega\sigma$$

$$i\omega\eta\sigma + (E_1 + E_2)\sigma = i\omega E_2\eta\varepsilon + E_1E_2\varepsilon \Rightarrow$$

$$\sigma = \underbrace{\frac{E_1 + i\omega\eta}{(E_1 + E_2) + i\omega\eta}E_2}_{E(i\omega)} \cdot \varepsilon$$

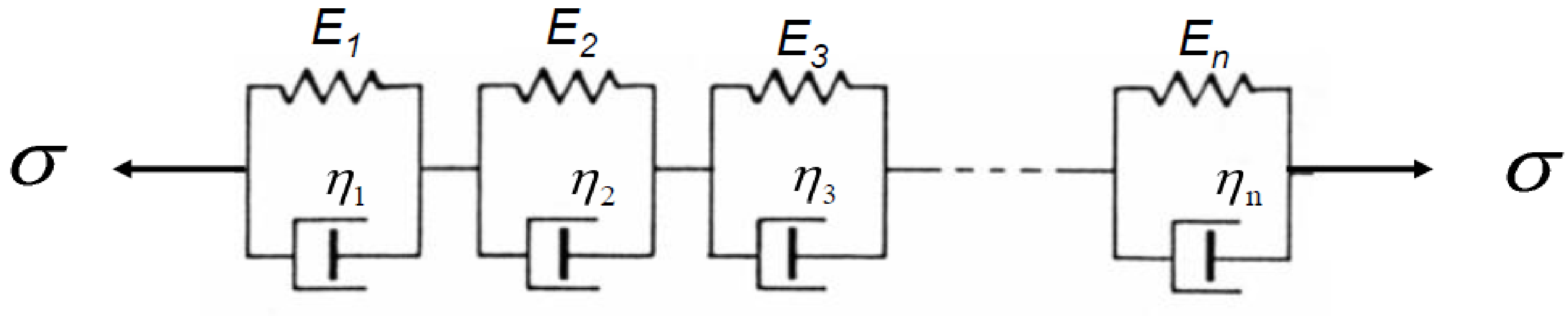
$$E(i\omega) = \frac{E_1 + i\omega\eta}{(E_1 + E_2) + i\omega\eta}E_2 \Rightarrow \ldots$$

$$E(i\omega) = \frac{E_1E_2(E_1 + E_2) + (\omega\eta)^2 E_2}{(E_1 + E_2)^2 + (\omega\eta)^2} + i\frac{\omega\eta E_2{}^2}{(E_1 + E_2)^2 + (\omega\eta)^2}$$

What if the curve of the model does not fit the curve of the material we want to describe?

Generalized Models…

Generalized Kelvin Model



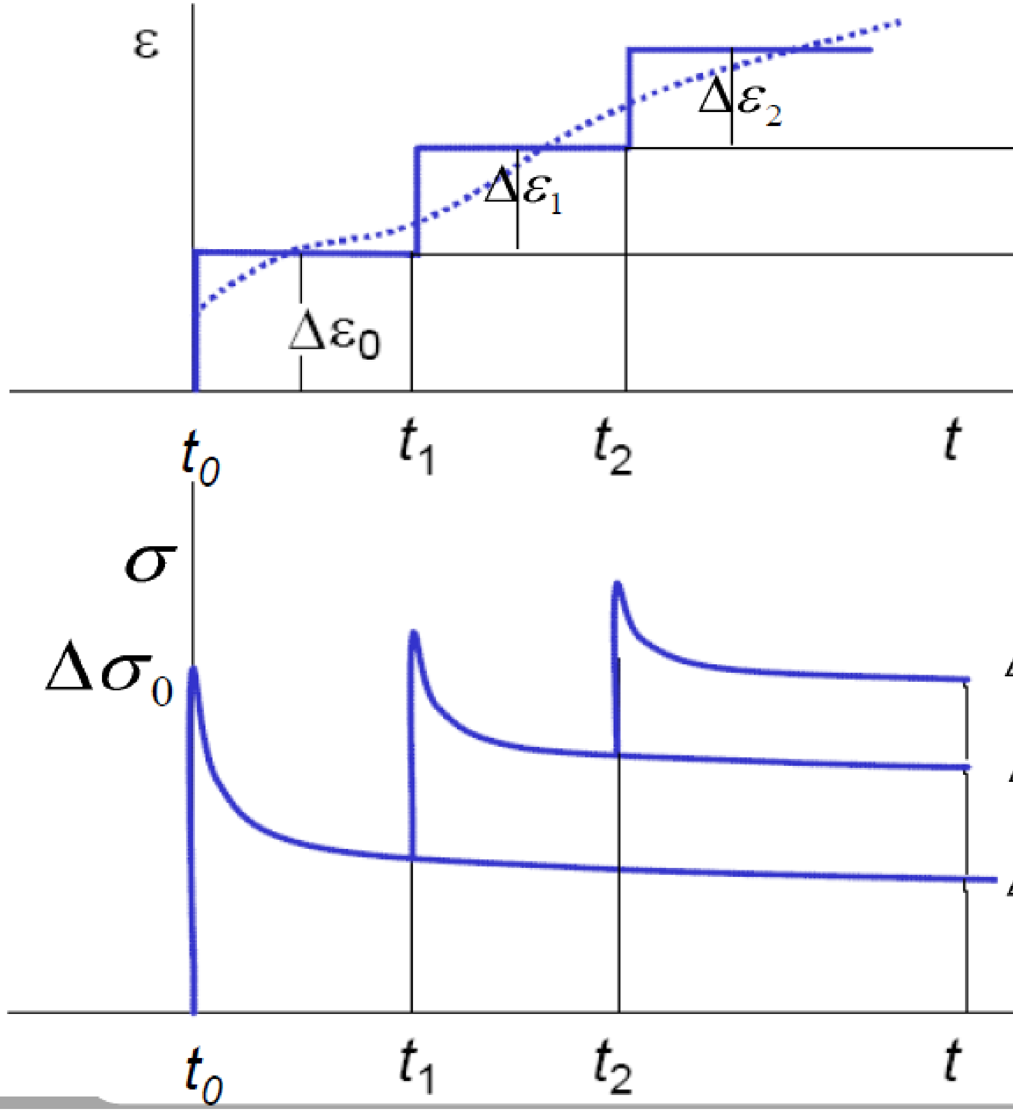
Total strain = $\sum$(strains in each Kelvin element)

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n = \sum_{i=1}^{n} \varepsilon_i$$

Creep function:

$$\varepsilon(t) = \sum_{i=1}^{n} \frac{\sigma_0}{E_i}\left(1 - e^{-\frac{E_i}{\eta_i}t}\right)$$

The Convolution Integral for Stress



$$\Delta\sigma_2 = \Delta\varepsilon_2 E(t - t_2)$$

$$\Delta\sigma_1 = \Delta\varepsilon_1 E(t - t_1)$$

$$\Delta\sigma_0 = \Delta\varepsilon_0 E(t - t_0)$$